APPENDIX

I

$$f(x) = \frac{\sin(x)}{x}$$

x	$\sin(x)/x$	x	$\sin(x)/x$	x	$\sin(x)/x$
0.0	1.00000	2.8	0.11964	5.6	−0.11273
0.1	0.99833	2.9	0.08250	5.7	−0.09661
0.2	0.99335	3.0	0.04704	5.8	−0.08010
0.3	0.98507	3.1	0.01341	5.9	−0.06337
0.4	0.97355	3.2	−0.01824	6.0	−0.04657
0.5	0.95885	3.3	−0.04780	6.1	−0.02986
0.6	0.94107	3.4	−0.07516	6.2	−0.01340
0.7	0.92031	3.5	−0.10022	6.3	0.00267
0.8	0.89670	3.6	−0.12292	6.4	0.01821
0.9	0.87036	3.7	−0.14320	6.5	0.03309
1.0	0.84147	3.8	−0.16101	6.6	0.04720
1.1	0.81019	3.9	−0.17635	6.7	0.06042
1.2	0.77670	4.0	−0.18920	6.8	0.07266
1.3	0.74120	4.1	−0.19958	6.9	0.08383
1.4	0.70389	4.2	−0.20752	7.0	0.09385
1.5	0.66500	4.3	−0.21306	7.1	0.10267
1.6	0.62473	4.4	−0.21627	7.2	0.11023
1.7	0.58333	4.5	−0.21723	7.3	0.11650
1.8	0.54103	4.6	−0.21602	7.4	0.12145
1.9	0.49805	4.7	−0.21275	7.5	0.12507
2.0	0.45465	4.8	−0.20753	7.6	0.12736
2.1	0.41105	4.9	−0.20050	7.7	0.12833
2.2	0.36750	5.0	−0.19179	7.8	0.12802
2.3	0.32422	5.1	−0.18153	7.9	0.12645
2.4	0.28144	5.2	−0.16990	8.0	0.12367
2.5	0.23939	5.3	−0.15703	8.1	0.11974
2.6	0.19827	5.4	−0.14310	8.2	0.11472
2.7	0.15829	5.5	−0.12828	8.3	0.10870

x	$\sin(x)/x$	x	$\sin(x)/x$	x	$\sin(x)/x$
8.4	0.10174	10.7	−0.08941	13.0	0.03232
8.5	0.09394	10.8	−0.09083	13.1	0.03883
8.6	0.08540	10.9	−0.09132	13.2	0.04485
8.7	0.07620	11.0	−0.09091	13.3	0.05034
8.8	0.06647	11.1	−0.08960	13.4	0.05525
8.9	0.05629	11.2	−0.08743	13.5	0.05954
9.0	0.04579	11.3	−0.08443	13.6	0.06317
9.1	0.03507	11.4	−0.08064	13.7	0.06613
9.2	0.02423	11.5	−0.07613	13.8	0.06838
9.3	0.01338	11.6	−0.07093	13.9	0.06993
9.4	0.00264	11.7	−0.06513	14.0	0.07076
9.5	−0.00791	11.8	−0.05877	14.1	0.07087
9.6	−0.01816	11.9	−0.05194	14.2	0.07028
9.7	−0.02802	12.0	−0.04471	14.3	0.06901
9.8	−0.03740	12.1	−0.03716	14.4	0.06706
9.9	−0.04622	12.2	−0.02936	14.5	0.06448
10.0	−0.05440	12.3	−0.02140	14.6	0.06129
10.1	−0.06189	12.4	−0.01336	14.7	0.05753
10.2	−0.06861	12.5	−0.00531	14.8	0.05326
10.3	−0.07453	12.6	0.00267	14.9	0.04852
10.4	−0.07960	12.7	0.01049	15.0	0.04335
10.5	−0.08378	12.8	0.01809		
10.6	−0.08705	12.9	0.02539		

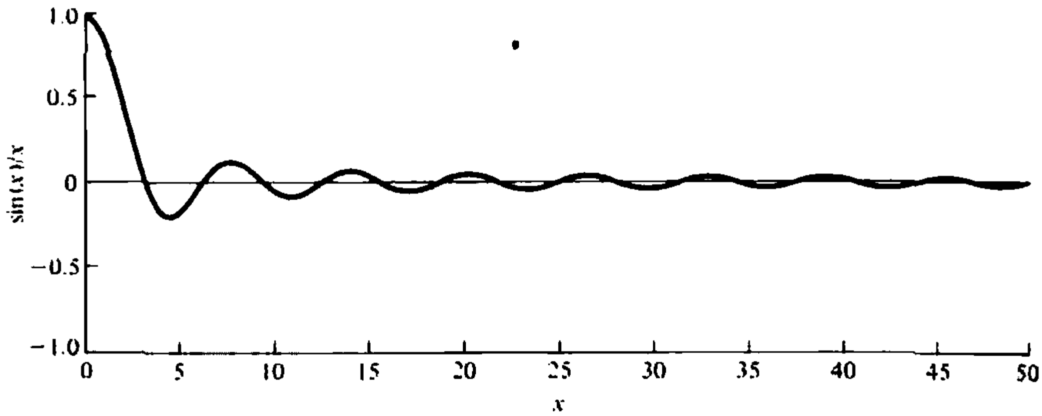


Figure I.1 Plot of sin $(x)/x$ function.

APPENDIX

II

$$f_N(x) = \left|\frac{\sin(Nx)}{N\sin(x)}\right|$$

$$N = 1, 3, 5, 10, 20$$

x	$f_1(x)$	$f_3(x)$	$f_5(x)$	$f_{10}(x)$	$f_{20}(x)$
0.0	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	1.00000	0.98671	0.96045	0.84287	0.45540
0.2	1.00000	0.94737	0.84711	0.45769	0.19047
0.3	1.00000	0.88356	0.67508	0.04775	0.04727
0.4	1.00000	0.79780	0.46700	0.19434	0.12703
0.5	1.00000	0.69353	0.24966	0.20001	0.05674
0.6	1.00000	0.57490	0.04998	0.04948	0.04751
0.7	1.00000	0.44664	0.10890	0.10198	0.07688
0.8	1.00000	0.31387	0.21100	0.13792	0.02007
0.9	1.00000	0.18186	0.24958	0.05261	0.04794
1.0	1.00000	0.05590	0.22792	0.06465	0.05425
1.1	1.00000	0.05900	0.15833	0.11221	0.00050
1.2	1.00000	0.15826	0.05996	0.05757	0.04858
1.3	1.00000	0.23793	0.04465	0.04361	0.03957
1.4	1.00000	0.29481	0.13334	0.10052	0.01375
1.5	1.00000	0.32666	0.18807	0.06519	0.04953
1.6	1.00000	0.33220	0.19796	0.02880	0.02758
1.7	1.00000	0.31120	0.16104	0.09695	0.02668
1.8	1.00000	0.26451	0.08464	0.07712	0.05092
1.9	1.00000	0.19398	0.01588	0.01584	0.01566
2.0	1.00000	0.10243	0.11966	0.10040	0.04097
2.1	1.00000	0.00649	0.20382	0.09692	0.05309
2.2	1.00000	0.12844	0.24737	0.00109	0.00109
2.3	1.00000	0.25856	0.23480	0.11348	0.06047
2.4	1.00000	0.39167	0.15888	0.13407	0.05687
2.5	1.00000	0.52244	0.02216	0.02211	0.02192
2.6	1.00000	0.64568	0.16301	0.14792	0.09570
2.7	1.00000	0.75646	0.37615	0.22378	0.06537
2.8	1.00000	0.85038	0.59143	0.08087	0.07785
2.9	1.00000	0.92368	0.78152	0.27738	0.20750
3.0	1.00000	0.97345	0.92161	0.70013	0.10799
3.1	1.00000	0.99769	0.99309	0.97172	0.88885
3.1415	1.00000	1.00000	1.00000	1.00000	1.00000

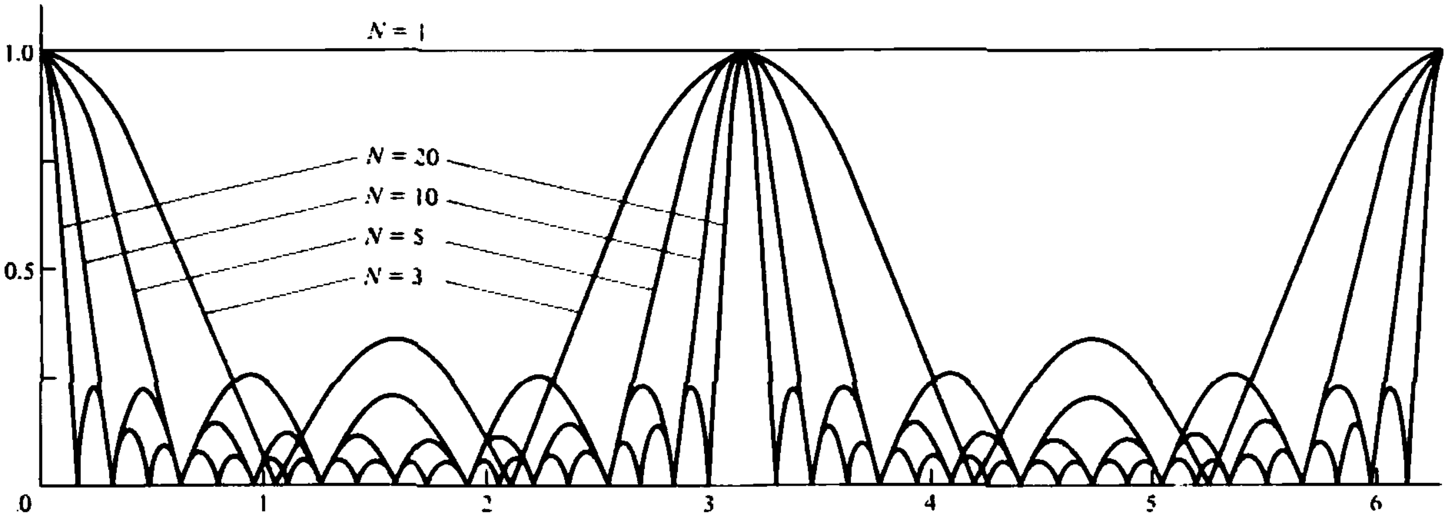


Figure II.1 Curves of $|\sin(Nx)/N\sin(x)|$ function.

APPENDIX

III

COSINE AND SINE INTEGRALS

$$S_i(x) = \int_0^x \frac{\sin(\tau)}{\tau}\, d\tau \tag{III-1}$$

$$C_i(x) = -\int_x^\infty \frac{\cos(\tau)}{\tau}\, d\tau = \int_\infty^x \frac{\cos(\tau)}{\tau}\, d\tau \tag{III-2}$$

$$C_{in}(x) = \int_0^x \frac{1 - \cos(\tau)}{\tau}\, d\tau \tag{III-3}$$

$$C_{in}(x) = \ln(\gamma x) - C_i(x) = \ln(\gamma) + \ln(x) - C_i(x)$$

$$C_{in}(x) = \ln(1.781) + \ln(x) - C_i(x) = 0.577215665 + \ln(x) - C_i(x) \tag{III-4}$$

Also

$$S_i(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}$$

$$C_i(x) = C + \ln(x) + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!}$$

$$C_{in}(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{2k(2k)!}$$

x	$S_i(x)$	$C_i(x)$	$C_{in}(x)$	x	$S_i(x)$	$C_i(x)$	$C_{in}(x)$
0.0	0.0	$-\infty$	0.00000	5.1	1.53125	−0.18348	2.38993
0.1	0.09994	−1.72787	0.00250	5.2	1.51367	−0.17525	2.40113
0.2	0.19956	−1.04221	0.00998	5.3	1.49732	−0.16551	2.41043
0.3	0.29850	−0.64917	0.02242	5.4	1.48230	−0.15439	2.41800
0.4	0.39646	−0.37881	0.03973	5.5	1.46872	−0.14205	2.42402
0.5	0.49311	−0.17778	0.06185	5.6	1.45667	−0.12867	2.42865
0.6	0.58813	−0.02227	0.08866	5.7	1.44620	−0.11441	2.43209
0.7	0.68122	0.10051	0.12003	5.8	1.43736	−0.09944	2.43451
0.8	0.77210	0.19828	0.15579	5.9	1.43018	−0.08393	2.43610
0.9	0.86047	0.27607	0.19579	6.0	1.42469	−0.06806	2.43703
1.0	0.94608	0.33740	0.23981	6.1	1.42087	−0.05198	2.43749
1.1	1.02869	0.38487	0.28765	6.2	1.41871	−0.03587	2.43764
1.2	1.10805	0.42046	0.33908	6.3	1.41817	−0.01989	2.43765
1.3	1.18396	0.44574	0.39384	6.4	1.41922	−0.00418	2.43770
1.4	1.25623	0.46201	0.45168	6.5	1.42179	0.01110	2.43792
1.5	1.32468	0.47036	0.51232	6.6	1.42582	0.02582	2.43846
1.6	1.38918	0.47173	0.57549	6.7	1.43121	0.03986	2.43947
1.7	1.44959	0.46697	0.64088	6.8	1.43787	0.05308	2.44106
1.8	1.50582	0.45681	0.70819	6.9	1.44570	0.06539	2.44334
1.9	1.55778	0.44194	0.77713	7.0	1.45460	0.07670	2.44643
2.0	1.60541	0.42298	0.84738	7.1	1.46443	0.08691	2.45040
2.1	1.64870	0.40051	0.91864	7.2	1.47509	0.09596	2.45534
2.2	1.68762	0.37507	0.99060	7.3	1.48644	0.10379	2.46130
2.3	1.72221	0.34718	1.06295	7.4	1.49834	0.11036	2.46834
2.4	1.75249	0.31729	1.13539	7.5	1.51068	0.11563	2.47649
2.5	1.77852	0.28587	1.20764	7.6	1.52331	0.11960	2.48577
2.6	1.80039	0.25334	1.27939	7.7	1.53611	0.12225	2.49619
2.7	1.81821	0.22008	1.35038	7.8	1.54894	0.12359	2.50775
2.8	1.83210	0.18649	1.42035	7.9	1.56167	0.12364	2.52044
2.9	1.84219	0.15290	1.48903	8.0	1.57419	0.12243	2.53422
3.0	1.84865	0.11963	1.55620	8.1	1.58637	0.12002	2.54906
3.1	1.85166	0.08699	1.62163	8.2	1.59810	0.11644	2.56491
3.2	1.85140	0.05526	1.68511	8.3	1.60928	0.11177	2.58170
3.3	1.84808	0.02468	1.74646	8.4	1.61981	0.10607	2.59938
3.4	1.84191	−0.00452	1.80551	8.5	1.62960	0.09943	2.61785
3.5	1.83313	−0.03213	1.86211	8.6	1.63857	0.09194	2.63704
3.6	1.82195	−0.05797	1.91612	8.7	1.64665	0.08368	2.65686
3.7	1.80862	−0.08190	1.96745	8.8	1.65379	0.07476	2.67721
3.8	1.79339	−0.10378	2.01599	8.9	1.65993	0.06528	2.69799
3.9	1.77650	−0.12350	2.06169	9.0	1.66504	0.05535	2.71909
4.0	1.75820	−0.14098	2.10449	9.1	1.66908	0.04507	2.74042
4.1	1.73874	−0.15617	2.14437	9.2	1.67205	0.03455	2.76186
4.2	1.71837	−0.16901	2.18131	9.3	1.67393	0.02391	2.78332
4.3	1.69732	−0.17951	2.21534	9.4	1.67473	0.01325	2.80467
4.4	1.67583	−0.18766	2.24648	9.5	1.67446	0.00268	2.82583
4.5	1.65414	−0.19349	2.27478	9.6	1.67316	−0.00771	2.84669
4.6	1.63246	−0.19705	2.30032	9.7	1.67084	−0.01780	2.86715
4.7	1.61101	−0.19839	2.32317	9.8	1.66757	−0.02752	2.88712
4.8	1.58998	−0.19760	2.34344	9.9	1.66338	−0.03676	2.90651
4.9	1.56956	−0.19478	2.36123	10.0	1.65835	−0.04546	2.92526
5.0	1.54993	−0.19003	2.37668	10.1	1.65253	−0.05352	2.94327

x	$S_i(x)$	$C_i(x)$	$C_{in}(x)$	x	$S_i(x)$	$C_i(x)$	$C_{in}(x)$
10.2	1.64600	−0.06089	2.96050	15.3	1.62865	0.02955	3.27552
10.3	1.63883	−0.06751	2.97687	15.4	1.63093	0.02345	3.28813
10.4	1.63112	−0.07332	2.99234	15.5	1.63258	0.01719	3.30086
10.5	1.62294	−0.07828	3.00687	15.6	1.63359	0.01084	3.31364
10.6	1.61439	−0.08237	3.02044	15.7	1.63396	0.00447	3.32641
10.7	1.60556	−0.08555	3.03301	15.8	1.63370	−0.00187	3.33910
10.8	1.59654	−0.08781	3.04457	15.9	1.63280	−0.00812	3.35165
10.9	1.58743	−0.08915	3.05513	16.0	1.63130	−0.01420	3.36400
11.0	1.57831	−0.08956	3.06467	16.1	1.62921	−0.02007	3.37610
11.1	1.56927	−0.08907	3.07323	16.2	1.62657	−0.02566	3.38789
11.2	1.56042	−0.08769	3.08082	16.3	1.62339	−0.03093	3.39931
11.3	1.55182	−0.08546	3.08748	16.4	1.61973	−0.03583	3.41033
11.4	1.54356	−0.08240	3.09323	16.5	1.61563	−0.04031	3.42088
11.5	1.53571	−0.07857	3.09813	16.6	1.61112	−0.04433	3.43095
11.6	1.52835	−0.07401	3.10224	16.7	1.60627	−0.04786	3.44049
11.7	1.52155	−0.06879	3.10560	16.8	1.60111	−0.05087	3.44947
11.8	1.51535	−0.06297	3.10828	16.9	1.59572	−0.05334	3.45787
11.9	1.50981	−0.05661	3.11036	17.0	1.59014	−0.05524	3.46567
12.0	1.50497	−0.04978	3.11190	17.1	1.58443	−0.05657	3.47287
12.1	1.50088	−0.04257	3.11299	17.2	1.57865	−0.05732	3.47945
12.2	1.49755	−0.03504	3.11369	17.3	1.57285	−0.05749	3.48541
12.3	1.49501	−0.02729	3.11410	17.4	1.56711	−0.05708	3.49076
12.4	1.49327	−0.01938	3.11429	17.5	1.56146	−0.05610	3.49552
12.5	1.49234	−0.01141	3.11435	17.6	1.55598	−0.05458	3.49969
12.6	1.49221	−0.00344	3.11436	17.7	1.55070	−0.05252	3.50330
12.7	1.49287	0.00443	3.11439	17.8	1.54568	−0.04997	3.50638
12.8	1.49430	0.01214	3.11452	17.9	1.54097	−0.04694	3.50895
12.9	1.49647	0.01961	3.11484	18.0	1.53661	−0.04348	3.51106
13.0	1.49936	0.02676	3.11540	18.1	1.53264	−0.03962	3.51274
13.1	1.50292	0.03355	3.11628	18.2	1.52909	−0.03540	3.51404
13.2	1.50711	0.03989	3.11754	18.3	1.52600	−0.03088	3.51500
13.3	1.51188	0.04574	3.11924	18.4	1.52339	−0.02610	3.51566
13.4	1.51716	0.05104	3.12143	18.5	1.52128	−0.02111	3.51609
13.5	1.52291	0.05576	3.12415	18.6	1.51969	−0.01596	3.51634
13.6	1.52905	0.05984	3.12744	18.7	1.51863	−0.01071	3.51644
13.7	1.53552	0.06327	3.13134	18.8	1.51810	−0.00540	3.51647
13.8	1.54225	0.06602	3.13587	18.9	1.51810	−0.00010	3.51648
13.9	1.54917	0.06806	3.14104	19.0	1.51863	0.00515	3.51650
14.0	1.55621	0.06940	3.14688	19.1	1.51967	0.01029	3.51661
14.1	1.56330	0.07002	3.15337	19.2	1.52122	0.01528	3.51685
14.2	1.57036	0.06993	3.16053	19.3	1.52324	0.02006	3.51726
14.3	1.57733	0.06914	3.16834	19.4	1.52572	0.02459	3.51790
14.4	1.58414	0.06767	3.17678	19.5	1.52863	0.02883	3.51880
14.5	1.59072	0.06554	3.18583	19.6	1.53192	0.03274	3.52000
14.6	1.59702	0.06278	3.19546	19.7	1.53557	0.03629	3.52155
14.7	1.60296	0.05943	3.20563	19.8	1.53954	0.03943	3.52347
14.8	1.60851	0.05554	3.21631	19.9	1.54378	0.04215	3.52579
14.9	1.61360	0.05113	3.22744	20.0	1.54824	0.04442	3.52853
15.0	1.61819	0.04628	3.23899				
15.1	1.62226	0.04102	3.25089				
15.2	1.62575	0.03543	3.26308				

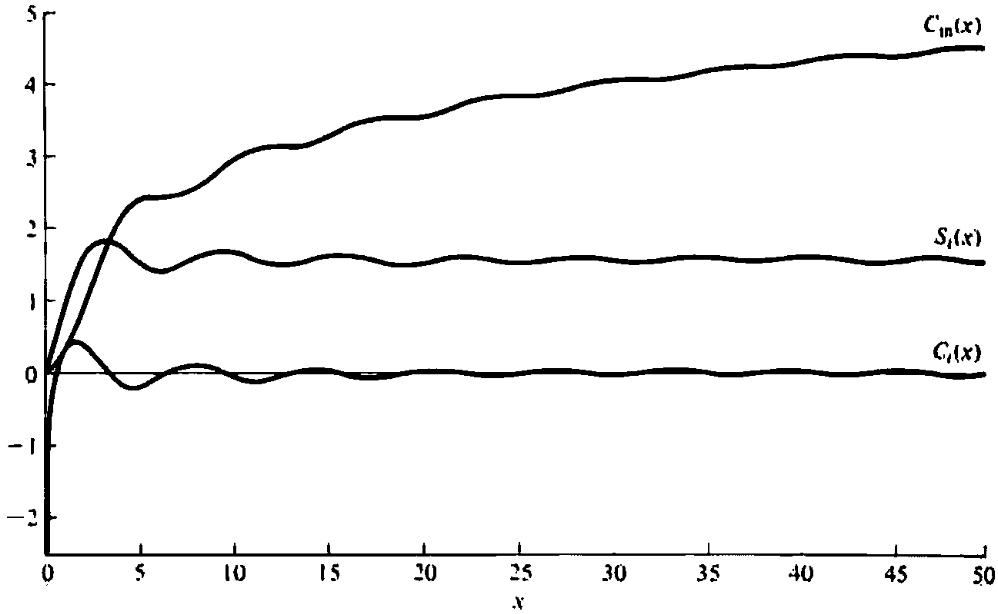


Figure III.1 Plots of sine and cosine integrals.

APPENDIX IV

FRESNEL INTEGRALS

$$C_0(x) = \int_0^x \frac{\cos(\tau)}{\sqrt{2\pi\tau}}\, d\tau \tag{IV-1}$$

$$S_0(x) = \int_0^x \frac{\sin(\tau)}{\sqrt{2\pi\tau}}\, d\tau \tag{IV-2}$$

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}\tau^2\right) d\tau \tag{IV-3}$$

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}\tau^2\right) d\tau \tag{IV-4}$$

$$C_1(x) = \int_x^\infty \cos(\tau^2)\, d\tau \tag{IV-5}$$

$$S_1(x) = \int_x^\infty \sin(\tau^2)\, d\tau \tag{IV-6}$$

$$C(x) - jS(x) = \int_0^x e^{-j(\pi/2)\tau^2}\, d\tau = \int_0^{(\pi/2)x^2} \frac{e^{-j\tau}}{\sqrt{2\pi\tau}}\, d\tau$$

$$C(x) - jS(x) = C_0\left(\frac{\pi}{2}x^2\right) - jS_0\left(\frac{\pi}{2}x^2\right) \tag{IV-7}$$

$$C_1(x) - jS_1(x) = \int_x^\infty e^{-j\tau^2}\, d\tau = \sqrt{\frac{\pi}{2}} \int_{x^2}^\infty \frac{e^{-j\tau}}{\sqrt{2\pi\tau}}\, d\tau$$

$$C_1(x) - jS_1(x) = \sqrt{\frac{\pi}{2}} \left\{ \int_0^\infty \frac{e^{-j\tau}}{\sqrt{2\pi\tau}}\, d\tau - \int_0^{x^2} \frac{e^{-j\tau}}{\sqrt{2\pi\tau}}\, d\tau \right\}$$

$$C_1(x) - jS_1(x) = \sqrt{\frac{\pi}{2}} \left\{ \left[\frac{1}{2} - j\frac{1}{2}\right] - \left[C_0(x^2) - jS_0(x^2)\right] \right\}$$

$$C_1(x) - jS_1(x) = \sqrt{\frac{\pi}{2}} \left\{ \left[\frac{1}{2} - C_0(x^2)\right] - j\left[\frac{1}{2} - S_0(x^2)\right] \right\} \tag{IV-8}$$

x	$C_1(x)$	$S_1(x)$	$C(x)$	$S(x)$
0.0	0.62666	0.62666	0.0	0.0
0.1	0.52666	0.62632	0.10000	0.00052
0.2	0.42669	0.62399	0.19992	0.00419
0.3	0.32690	0.61766	0.29940	0.01412
0.4	0.22768	0.60536	0.39748	0.03336
0.5	0.12977	0.58518	0.49234	0.06473
0.6	0.03439	0.55532	0.58110	0.11054
0.7	−0.05672	0.51427	0.65965	0.17214
0.8	−0.14119	0.46092	0.72284	0.24934
0.9	−0.21606	0.39481	0.76482	0.33978
1.0	−0.27787	0.31639	0.77989	0.43826
1.1	−0.32285	0.22728	0.76381	0.53650
1.2	−0.34729	0.13054	0.71544	0.62340
1.3	−0.34803	0.03081	0.63855	0.68633
1.4	−0.32312	−0.06573	0.54310	0.71353
1.5	−0.27253	−0.15158	0.44526	0.69751
1.6	−0.19886	−0.21861	0.36546	0.63889
1.7	−0.10790	−0.25905	0.32383	0.54920
1.8	−0.00871	−0.26682	0.33363	0.45094
1.9	0.08680	−0.23918	0.39447	0.37335
2.0	0.16520	−0.17812	0.48825	0.34342
2.1	0.21359	−0.09141	0.58156	0.37427
2.2	0.22242	0.00743	0.63629	0.45570
2.3	0.18833	0.10054	0.62656	0.55315
2.4	0.11650	0.16879	0.55496	0.61969
2.5	0.02135	0.19614	0.45742	0.61918
2.6	−0.07518	0.17454	0.38894	0.54999
2.7	−0.14816	0.10789	0.39249	0.45292
2.8	−0.17646	0.01329	0.46749	0.39153
2.9	−0.15021	−0.08181	0.56237	0.41014
3.0	−0.07621	−0.14690	0.60572	0.49631
3.1	0.02152	−0.15883	0.56160	0.58181
3.2	0.10791	−0.11181	0.46632	0.59335
3.3	0.14907	−0.02260	0.40570	0.51929
3.4	0.12691	0.07301	0.43849	0.42965
3.5	0.04965	0.13335	0.53257	0.41525
3.6	−0.04819	0.12973	0.58795	0.49231
3.7	−0.11929	0.06258	0.54195	0.57498
3.8	−0.12649	−0.03483	0.44810	0.56562
3.9	−0.06469	−0.11030	0.42233	0.47521
4.0	0.03219	−0.12048	0.49842	0.42052
4.1	0.10690	−0.05815	0.57369	0.47580
4.2	0.11228	0.03885	0.54172	0.56320
4.3	0.04374	0.10751	0.44944	0.55400
4.4	−0.05287	0.10038	0.43833	0.46227
4.5	−0.10884	0.02149	0.52602	0.43427
4.6	−0.08188	−0.07126	0.56724	0.51619
4.7	0.00810	−0.10594	0.49143	0.56715
4.8	0.08905	−0.05381	0.43380	0.49675
4.9	0.09277	0.04224	0.50016	0.43507
5.0	0.01519	0.09874	0.56363	0.49919
5.1	−0.07411	0.06405	0.49979	0.56239
5.2	−0.09125	−0.03004	0.43889	0.49688

x	$C_1(x)$	$S_1(x)$	$C(x)$	$S(x)$
5.3	−0.01892	−0.09235	0.50778	0.44047
5.4	0.07063	−0.05976	0.55723	0.51403
5.5	0.08408	0.03440	0.47843	0.55369
5.6	0.00641	0.08900	0.45171	0.47004
5.7	−0.07642	0.04296	0.53846	0.45953
5.8	−0.06919	−0.05135	0.52984	0.54604
5.9	0.01998	−0.08231	0.44859	0.51633
6.0	0.08245	−0.01181	0.49953	0.44696
6.1	0.03946	0.07180	0.54950	0.51647
6.2	−0.05363	0.06018	0.46761	0.53982
6.3	−0.07284	−0.03144	0.47600	0.45555
6.4	0.00835	−0.07765	0.54960	0.49649
6.5	0.07574	−0.01326	0.48161	0.54538
6.6	0.03183	0.06872	0.46899	0.46307
6.7	−0.05828	0.04658	0.54674	0.49150
6.8	−0.05734	−0.04600	0.48307	0.54364
6.9	0.03317	−0.06440	0.47322	0.46244
7.0	0.06832	0.02077	0.54547	0.49970
7.1	−0.00944	0.06977	0.47332	0.53602
7.2	−0.06943	0.00041	0.48874	0.45725
7.3	−0.00864	−0.06793	0.53927	0.51894
7.4	0.06582	−0.01521	0.46010	0.51607
7.5	0.02018	0.06353	0.51601	0.46070
7.6	−0.06137	0.02367	0.51564	0.53885
7.7	−0.02580	−0.05958	0.46278	0.48202
7.8	0.05828	−0.02668	0.53947	0.48964
7.9	0.02638	0.05752	0.47598	0.53235
8.0	−0.05730	0.02494	0.49980	0.46021
8.1	−0.02238	−0.05752	0.52275	0.53204
8.2	0.05803	−0.01870	0.46384	0.48589
8.3	0.01387	0.05861	0.53775	0.49323
8.4	−0.05899	0.00789	0.47092	0.52429
8.5	−0.00080	−0.05881	0.51417	0.46534
8.6	0.05767	0.00729	0.50249	0.53693
8.7	−0.01616	0.05515	0.48274	0.46774
8.8	−0.05079	−0.02545	0.52797	0.52294
8.9	0.03461	−0.04425	0.46612	0.48856
9.0	0.03526	0.04293	0.53537	0.49985
9.1	−0.04951	0.02381	0.46661	0.51042
9.2	−0.01021	−0.05338	0.52914	0.48135
9.3	0.05354	0.00485	0.47628	0.52467
9.4	−0.02020	0.04920	0.51803	0.47134
9.5	−0.03995	−0.03426	0.48729	0.53100
9.6	0.04513	−0.02599	0.50813	0.46786
9.7	0.00837	0.05086	0.49549	0.53250
9.8	−0.04983	−0.01094	0.50192	0.46758
9.9	0.02916	−0.04124	0.49961	0.53215
10.0	0.02554	0.04298	0.49989	0.46817
10.1	−0.04927	0.00478	0.49961	0.53151
10.2	0.01738	−0.04583	0.50186	0.46885
10.3	0.03233	0.03621	0.49575	0.53061
10.4	−0.04681	0.01094	0.50751	0.47033
10.5	0.01360	−0.04563	0.48849	0.52804

x	$C_1(x)$	$S_1(x)$	$C(x)$	$S(x)$
10.6	0.03187	0.03477	0.51601	0.47460
10.7	−0.04595	0.00848	0.47936	0.52143
10.8	0.01789	−0.04270	0.52484	0.48413
10.9	0.02494	0.03850	0.47211	0.50867
11.0	−0.04541	−0.00202	0.52894	0.49991
11.1	0.02845	−0.03492	0.47284	0.49079
11.2	0.01008	0.04349	0.52195	0.51805
11.3	−0.03981	−0.01930	0.48675	0.47514
11.4	0.04005	−0.01789	0.50183	0.52786
11.5	−0.01282	0.04155	0.51052	0.47440
11.6	−0.02188	−0.03714	0.47890	0.51755
11.7	0.04164	0.00962	0.52679	0.49525
11.8	−0.03580	0.02267	0.47489	0.49013
11.9	0.00977	−0.04086	0.51544	0.52184
12.0	0.02059	0.03622	0.49993	0.47347
12.1	−0.03919	−0.01309	0.48426	0.52108
12.2	0.03792	−0.01555	0.52525	0.49345
12.3	−0.01914	0.03586	0.47673	0.48867
12.4	−0.00728	−0.03966	0.50951	0.52384
12.5	0.02960	0.02691	0.50969	0.47645
12.6	−0.03946	−0.00421	0.47653	0.50936
12.7	0.03445	−0.01906	0.52253	0.51097
12.8	−0.01783	0.03475	0.49376	0.47593
12.9	−0.00377	−0.03857	0.48523	0.51977
13.0	0.02325	0.03064	0.52449	0.49994
13.1	−0.03530	−0.01452	0.48598	0.48015
13.2	0.03760	−0.00459	0.49117	0.52244
13.3	−0.03075	0.02163	0.52357	0.49583
13.4	0.01744	−0.03299	0.48482	0.48173
13.5	−0.00129	0.03701	0.49103	0.52180
13.6	−0.01421	−0.03391	0.52336	0.49848
13.7	0.02639	0.02521	0.48908	0.47949
13.8	−0.03377	−0.01313	0.48534	0.51781
13.9	0.03597	−0.00002	0.52168	0.50737
14.0	−0.03352	0.01232	0.49996	0.47726
14.1	0.02749	−0.02240	0.47844	0.50668
14.2	−0.01916	0.02954	0.51205	0.51890
14.3	0.00979	−0.03357	0.51546	0.48398
14.4	−0.00043	0.03472	0.48131	0.48819
14.5	−0.00817	−0.03350	0.49164	0.52030
14.6	0.01553	0.03052	0.52113	0.50538
14.7	−0.02145	−0.02640	0.50301	0.47856
14.8	0.02591	0.02168	0.47853	0.49869
14.9	−0.02903	−0.01683	0.49971	0.52136
15.0	0.03103	0.01217	0.52122	0.49926

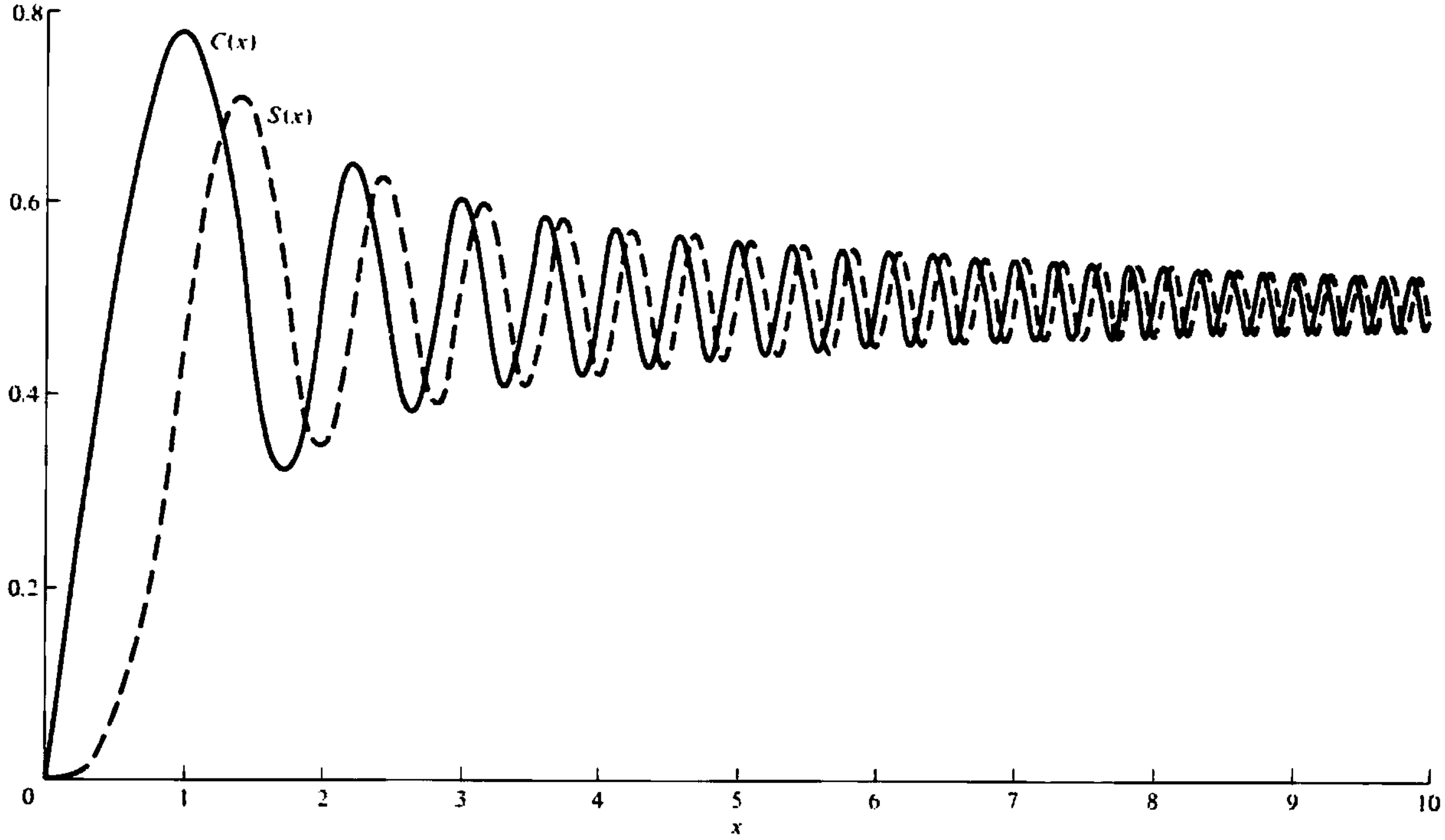


Figure IV.1 Plots of $C(x)$ and $S(x)$ Fresnel integrals.

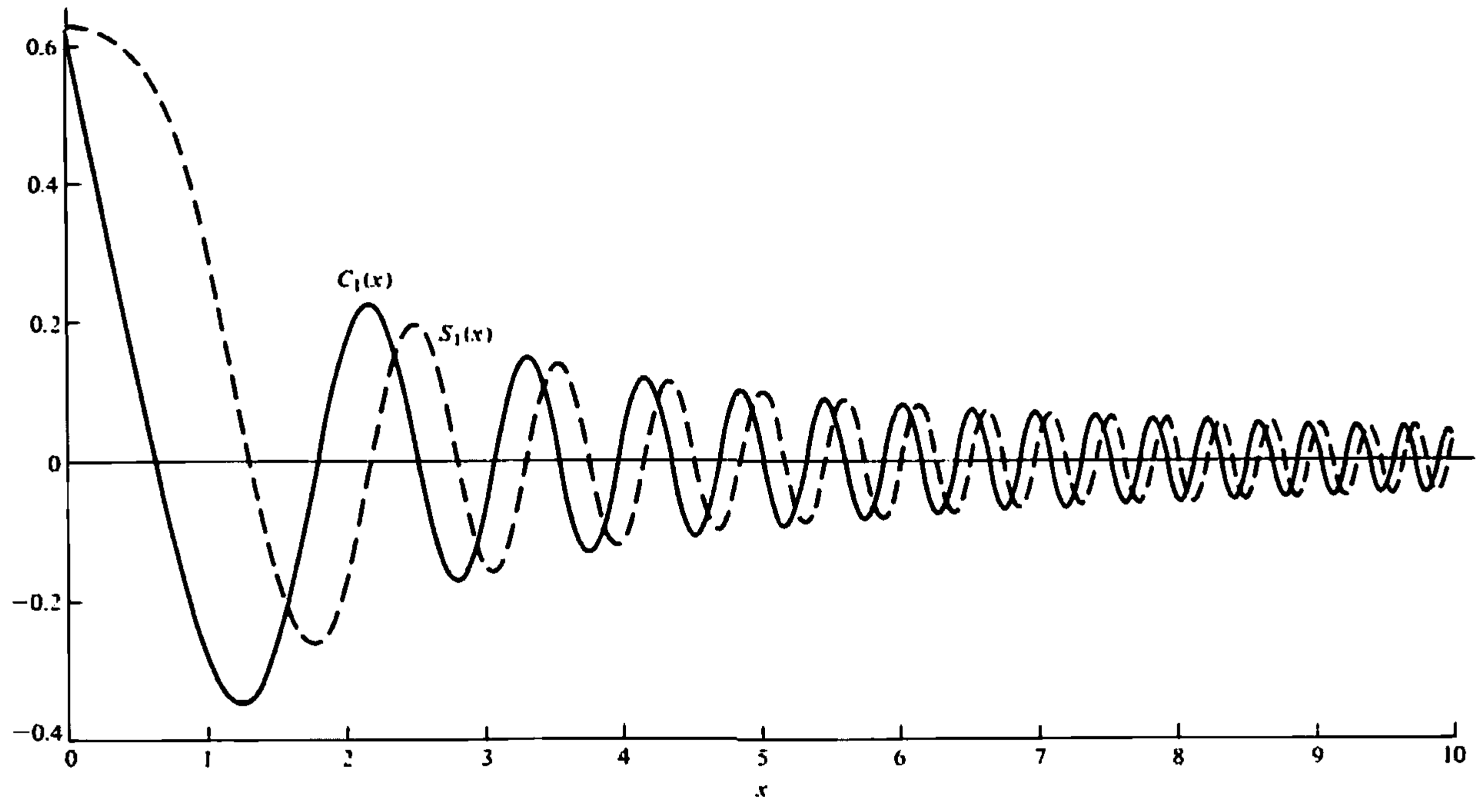


Figure IV.2 Plots of $C_1(x)$ and $S_1(x)$ Fresnel integrals.

APPENDIX

V

BESSEL FUNCTIONS

Bessel's equation can be written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0 \tag{V-1}$$

Using the method of Frobenius, we can write its solutions as

$$y(x) = A_1 J_p(x) + B_1 J_{-p}(x), \qquad p \neq 0 \text{ or integer} \tag{V-2}$$

or

$$y(x) = A_2 J_n(x) + B_2 Y_n(x), \qquad p = n = 0 \text{ or integer} \tag{V-3}$$

where

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+p}}{m!(m+p)!} \tag{V-4}$$

$$J_{-p}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m-p}}{m!(m-p)!} \tag{V-5}$$

$$Y_p(x) = \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \tag{V-6}$$

$$m! = \Gamma(m+1) \tag{V-7}$$

$J_p(x)$ is referred to as the Bessel function of the first kind of order p, $Y_p(x)$ as the Bessel function of the second kind of order p, and $\Gamma(x)$ as the gamma function.

When $p = n =$ integer, using (V-5) and (V-7) it can be shown that

$$J_{-n}(x) = (-1)^n J_n(x) \tag{V-8}$$

and no longer are the two Bessel functions independent of each other. Therefore a second solution is required and it is given by (V-3). It can also be shown that

$$Y_n(x) = \lim_{p\to n} Y_p(x) = \lim_{p\to n} \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \tag{V-9}$$

When the argument of the Bessel function is negative and $p = n$, using (V-4) leads to

$$J_n(-x) = (-1)^n J_n(x) \tag{V-10}$$

In many applications Bessel functions of small and large arguments are required. Using asymptotic methods, it can be shown that

$$\left.\begin{aligned} &J_0(x) \simeq 1 \\ &Y_0(x) \simeq \frac{2}{\pi}\ln\left(\frac{\gamma x}{2}\right) \\ &\gamma = 1.781 \end{aligned}\right\} x \to 0 \tag{V-11}$$

$$\left.\begin{aligned} &J_p(x) \simeq \frac{1}{p!}\left(\frac{x}{2}\right)^p \\ &Y_p(x) \simeq -\frac{(p-1)!}{\pi}\left(\frac{2}{x}\right)^p \end{aligned}\right\} \begin{aligned} &x \to 0 \\ &p > 0 \end{aligned} \tag{V-12}$$

and

$$\left.\begin{aligned} &J_p(x) \simeq \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right) \\ &Y_p(x) \simeq \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right) \end{aligned}\right\} x \to \infty \tag{V-13}$$

For wave propagation it is often convenient to introduce Hankel functions defined as

$$H_p^{(1)}(x) = J_p(x) + jY_p(x) \tag{V-14}$$

$$H_p^{(2)}(x) = J_p(x) - jY_p(x) \tag{V-15}$$

where $H_p^{(1)}(x)$ is the Hankel function of the first kind of order p and $H_p^{(2)}(x)$ is the Hankel function of the second kind of order p. For large arguments

$$H_p^{(1)}(x) \simeq \sqrt{\frac{2}{\pi x}}\, e^{j[x - p(\pi/2) - \pi/4]}, \qquad x \to \infty \tag{V-16}$$

$$H_p^{(2)}(x) \simeq \sqrt{\frac{2}{\pi x}}\, e^{-j[x - p(\pi/2) - \pi/4]}, \qquad x \to \infty \tag{V-17}$$

A derivative can be taken using either

$$\frac{d}{dx}[Z_p(\alpha x)] = \alpha Z_{p-1}(\alpha x) - \frac{p}{x}Z_p(\alpha x) \tag{V-18}$$

or

$$\frac{d}{dx}[Z_p(\alpha x)] = -\alpha Z_{p+1}(\alpha x) + \frac{p}{x}Z_p(\alpha x) \tag{V-19}$$

where Z_p can be a Bessel function (J_p, Y_p) or a Hankel function [$H_p^{(1)}$ or $H_p^{(2)}$].

A useful identity relating Bessel functions and their derivatives is given by

$$J_p(x)\,Y_p'(x) - Y_p(x)\,J_p'(x) = \frac{2}{\pi x} \tag{V-20}$$

and it is referred to as the Wronskian. The prime (′) indicates a derivative. Also

$$J_p(x)\,J'_{-p}(x) - J_{-p}(x)\,J_p'(x) = -\frac{2}{\pi x}\sin(p\pi) \tag{V-21}$$

Some useful integrals of Bessel functions are

$$\int x^{p+1} J_p(\alpha x)\,dx = \frac{1}{\alpha}x^{p+1}J_{p+1}(\alpha x) + C \tag{V-22}$$

$$\int x^{1-p} J_p(\alpha x)\,dx = -\frac{1}{\alpha}x^{1-p}J_{p-1}(\alpha x) + C \tag{V-23}$$

$$\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C \tag{V-24}$$

$$\int x^6 J_1(x)\,dx = x^6 J_2(x) - 4x^5 J_3(x) + 8x^4 J_4(x) + C \tag{V-25}$$

$$\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x) + C \tag{V-26}$$

$$\int xJ_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx + C \tag{V-27}$$

$$\int x^{-1} J_1(x)\,dx = -J_1(x) + \int J_0(x)\,dx + C \tag{V-28}$$

$$\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx + C \tag{V-29}$$

$$\int x^m J_n(x)\,dx = x^m J_{n+1}(x) - (m - n - 1)\int x^{m-1} J_{n+1}(x)\,dx \tag{V-30}$$

$$\int x^m J_n(x)\,dx = -x^m J_{n-1}(x) + (m + n - 1)\int x^{m-1} J_{n-1}(x)\,dx \tag{V-31}$$

$$J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin\theta)\sin\theta\,d\theta \tag{V-32}$$

$$\frac{1}{x}J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\cos^2\theta\,d\theta \tag{V-33}$$

$$J_2(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\cos 2\theta\,d\theta \tag{V-34}$$

$$J_n(x) = \frac{j^{-n}}{2\pi}\int_0^{2\pi} e^{jx\cos\phi}e^{jn\phi}\,d\phi \tag{V-35}$$

$$J_n(x) = \frac{j^{-n}}{\pi} \int_0^{\pi} \cos(n\phi)\, e^{jx\cos\phi}\, d\phi \tag{V-36}$$

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\phi - n\phi)\, d\phi \tag{V-37}$$

$$J_{2n}(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin\phi) \cos(2n\phi)\, d\phi \tag{V-38}$$

$$J_{2n}(x) = (-1)^n \frac{2}{\pi} \int_0^{\pi/2} \cos(x \cos\phi) \cos(2n\phi)\, d\phi \tag{V-39}$$

The integrals

$$\int_0^x J_0(\tau)\, d\tau \quad \text{and} \quad \int_0^x Y_0(\tau)\, d\tau \tag{V-40}$$

often appear in solutions of problems but cannot be integrated in closed form. Graphs and tables for each, obtained using numerical techniques, are included.

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0.0	1.00000	0.0	$-\infty$	$-\infty$
0.1	0.99750	0.04994	−1.53424	−6.45895
0.2	0.99003	0.09950	−1.08110	−3.32382
0.3	0.97763	0.14832	−0.80727	−2.29310
0.4	0.96040	0.19603	−0.60602	−1.78087
0.5	0.93847	0.24227	−0.44452	−1.47147
0.6	0.91201	0.28670	−0.30851	−1.26039
0.7	0.88120	0.32900	−0.19066	−1.10325
0.8	0.84629	0.36884	−0.08680	−0.97814
0.9	0.80752	0.40595	0.00563	−0.87313
1.0	0.76520	0.44005	0.08826	−0.78121
1.1	0.71962	0.47090	0.16216	−0.69812
1.2	0.67113	0.49829	0.22808	−0.62114
1.3	0.62009	0.52202	0.28654	−0.54852
1.4	0.56686	0.54195	0.33789	−0.47915
1.5	0.51183	0.55794	0.38245	−0.41231
1.6	0.45540	0.56990	0.42043	−0.34758
1.7	0.39799	0.57777	0.45203	−0.28473
1.8	0.33999	0.58152	0.47743	−0.22366
1.9	0.28182	0.58116	0.49682	−0.16441
2.0	0.22389	0.57673	0.51038	−0.10703
2.1	0.16661	0.56829	0.51829	−0.05168
2.2	0.11036	0.55596	0.52078	0.00149
2.3	0.05554	0.53987	0.51807	0.05228
2.4	0.00251	0.52019	0.51041	0.10049
2.5	−0.04838	0.49710	0.49807	0.14592
2.6	−0.09681	0.47082	0.48133	0.18836
2.7	−0.14245	0.44161	0.46050	0.22763
2.8	−0.18504	0.40972	0.43592	0.26354
2.9	−0.22432	0.37544	0.40791	0.29594
3.0	−0.26005	0.33906	0.37686	0.32467
3.1	−0.29206	0.30092	0.34310	0.34963
3.2	−0.32019	0.26134	0.30705	0.37071
3.3	−0.34430	0.22066	0.26909	0.38785
3.4	−0.36430	0.17923	0.22962	0.40101
3.5	−0.38013	0.13738	0.18902	0.41019
3.6	−0.39177	0.09547	0.14771	0.41539
3.7	−0.39923	0.05383	0.10607	0.41667
3.8	−0.40256	0.01282	0.06450	0.41411
3.9	−0.40183	−0.02724	0.02338	0.40782
4.0	−0.39715	−0.06604	−0.01694	0.39793
4.1	−0.38868	−0.10328	−0.05609	0.38459
4.2	−0.37657	−0.13865	−0.09375	0.36801
4.3	−0.36102	−0.17190	−0.12960	0.34839
4.4	−0.34226	−0.20278	−0.16334	0.32597
4.5	−0.32054	−0.23106	−0.19471	0.30100
4.6	−0.29614	−0.25655	−0.22346	0.27375
4.7	−0.26933	−0.27908	−0.24939	0.24450
4.8	−0.24043	−0.29850	−0.27230	0.21356
4.9	−0.20974	−0.31470	−0.29205	0.18125
5.0	−0.17760	−0.32758	−0.30852	0.14786

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
5.1	−0.14434	−0.33710	−0.32160	0.11374
5.2	−0.11029	−0.34322	−0.33125	0.07919
5.3	−0.07580	−0.34596	−0.33744	0.04455
5.4	−0.04121	−0.34534	−0.34017	0.01013
5.5	−0.00684	−0.34144	−0.33948	−0.02376
5.6	0.02697	−0.33433	−0.33544	−0.05681
5.7	0.05992	−0.32415	−0.32816	−0.08872
5.8	0.09170	−0.31103	−0.31775	−0.11923
5.9	0.12203	−0.29514	−0.30437	−0.14808
6.0	0.15065	−0.27668	−0.28819	−0.17501
6.1	0.17729	−0.25587	−0.26943	−0.19981
6.2	0.20175	−0.23292	−0.24831	−0.22228
6.3	0.22381	−0.20809	−0.22506	−0.24225
6.4	0.24331	−0.18164	−0.19995	−0.25956
6.5	0.26009	−0.15384	−0.17324	−0.27409
6.6	0.27404	−0.12498	−0.14523	−0.28575
6.7	0.28506	−0.09534	−0.11619	−0.29446
6.8	0.29310	−0.06522	−0.08643	−0.30019
6.9	0.29810	−0.03490	−0.05625	−0.30292
7.0	0.30008	−0.00468	−0.02595	−0.30267
7.1	0.29905	+0.02515	0.00418	−0.29948
7.2	0.29507	0.05433	0.03385	−0.29342
7.3	0.28822	0.08257	0.06277	−0.28459
7.4	0.27860	0.10962	0.09068	−0.27311
7.5	0.26634	0.13525	0.11731	−0.25913
7.6	0.25160	0.15921	0.14243	−0.24280
7.7	0.23456	0.18131	0.16580	−0.22432
7.8	0.21541	0.20136	0.18723	−0.20388
7.9	0.19436	0.21918	0.20652	−0.18172
8.0	0.17165	0.23464	0.22352	−0.15806
8.1	0.14752	0.24761	0.23809	−0.13315
8.2	0.12222	0.25800	0.25012	−0.10724
8.3	0.09601	0.26574	0.25951	−0.08060
8.4	0.06916	0.27079	0.26622	−0.05348
8.5	0.04194	0.27312	0.27021	−0.02617
8.6	0.01462	0.27276	0.27146	0.00108
8.7	−0.01252	0.26972	0.27000	0.02801
8.8	−0.03923	0.26407	0.26587	0.05436
8.9	−0.06525	0.25590	0.25916	0.07987
9.0	−0.09033	0.24531	0.24994	0.10431
9.1	−0.11424	0.23243	0.23834	0.12747
9.2	−0.13675	0.21741	0.22449	0.14911
9.3	−0.15765	0.20041	0.20857	0.16906
9.4	−0.17677	0.18163	0.19074	0.18714
9.5	−0.19393	0.16126	0.17121	0.20318
9.6	−0.20898	0.13952	0.15018	0.21706
9.7	−0.22180	0.11664	0.12787	0.22866
9.8	−0.23228	0.09284	0.10453	0.23789
9.9	−0.24034	0.06837	0.08038	0.24469
10.0	−0.24594	0.04347	0.05567	0.24902

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
10.1	−0.24903	0.01840	0.03066	0.25084
10.2	−0.24962	−0.00662	0.00558	0.25019
10.3	−0.24772	−0.03132	−0.01930	0.24707
10.4	−0.24337	−0.05547	−0.04375	0.24155
10.5	−0.23665	−0.07885	−0.06753	0.23370
10.6	−0.22764	−0.10123	−0.09042	0.22363
10.7	−0.21644	−0.12240	−0.11219	0.21144
10.8	−0.20320	−0.14217	−0.13264	0.19729
10.9	−0.18806	−0.16035	−0.15158	0.18132
11.0	−0.17119	−0.17679	−0.16885	0.16371
11.1	−0.15277	−0.19133	−0.18428	0.14464
11.2	−0.13299	−0.20385	−0.19773	0.12431
11.3	−0.11207	−0.21426	−0.20910	0.10294
11.4	−0.09021	−0.22245	−0.21829	0.08074
11.5	−0.06765	−0.22838	−0.22523	0.05794
11.6	−0.04462	−0.23200	−0.22987	0.03477
11.7	−0.02133	−0.23330	−0.23218	0.01145
11.8	0.00197	−0.23229	−0.23216	−0.01179
11.9	0.02505	−0.22898	−0.22983	−0.03471
12.0	0.04769	−0.22345	−0.22524	−0.05710
12.1	0.06967	−0.21575	−0.21844	−0.07874
12.2	0.09077	−0.20598	−0.20952	−0.09942
12.3	0.11080	−0.19426	−0.19859	−0.11895
12.4	0.12956	−0.18071	−0.18578	−0.13714
12.5	0.14689	−0.16549	−0.17121	−0.15384
12.6	0.16261	−0.14874	−0.15506	−0.16888
12.7	0.17659	−0.13066	−0.13750	−0.18213
12.8	0.18870	−0.11143	−0.11870	−0.19347
12.9	0.19885	−0.09125	−0.09887	−0.20282
13.0	0.20693	−0.07032	−0.07821	−0.21008
13.1	0.21289	−0.04885	−0.05692	−0.21521
13.2	0.21669	−0.02707	−0.03524	−0.21817
13.3	0.21830	−0.00518	−0.01336	−0.21895
13.4	0.21773	0.01660	0.00848	−0.21756
13.5	0.21499	0.03805	0.03008	−0.21402
13.6	0.21013	0.05896	0.05122	−0.20839
13.7	0.20322	0.07914	0.07169	−0.20074
13.8	0.19434	0.09839	0.09130	−0.19116
13.9	0.18358	0.11653	0.10986	−0.17975
14.0	0.17108	0.13338	0.12719	−0.16664
14.1	0.15695	0.14879	0.14314	−0.15198
14.2	0.14137	0.16261	0.15754	−0.13592
14.3	0.12449	0.17473	0.17028	−0.11862
14.4	0.10649	0.18503	0.18123	−0.10026
14.5	0.08755	0.19343	0.19030	−0.08104
14.6	0.06787	0.19986	0.19742	−0.06115
14.7	0.04764	0.20426	0.20252	−0.04079
14.8	0.02708	0.20660	0.20557	−0.02016
14.9	0.00639	0.20688	0.20655	0.00053
15.0	−0.01422	0.20511	0.20546	0.02107

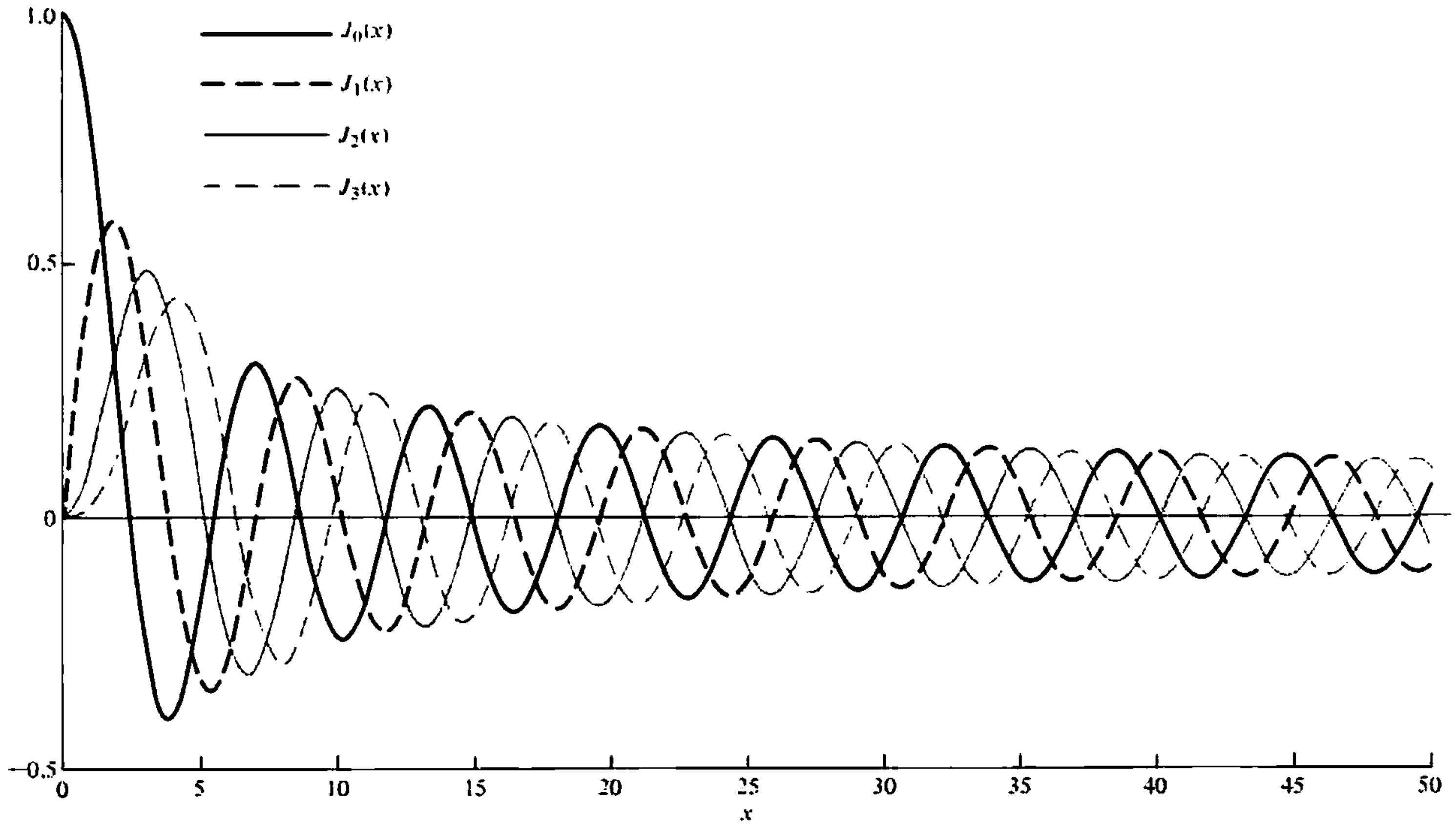


Figure V.1 Bessel functions of the first kind [$J_0(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$].

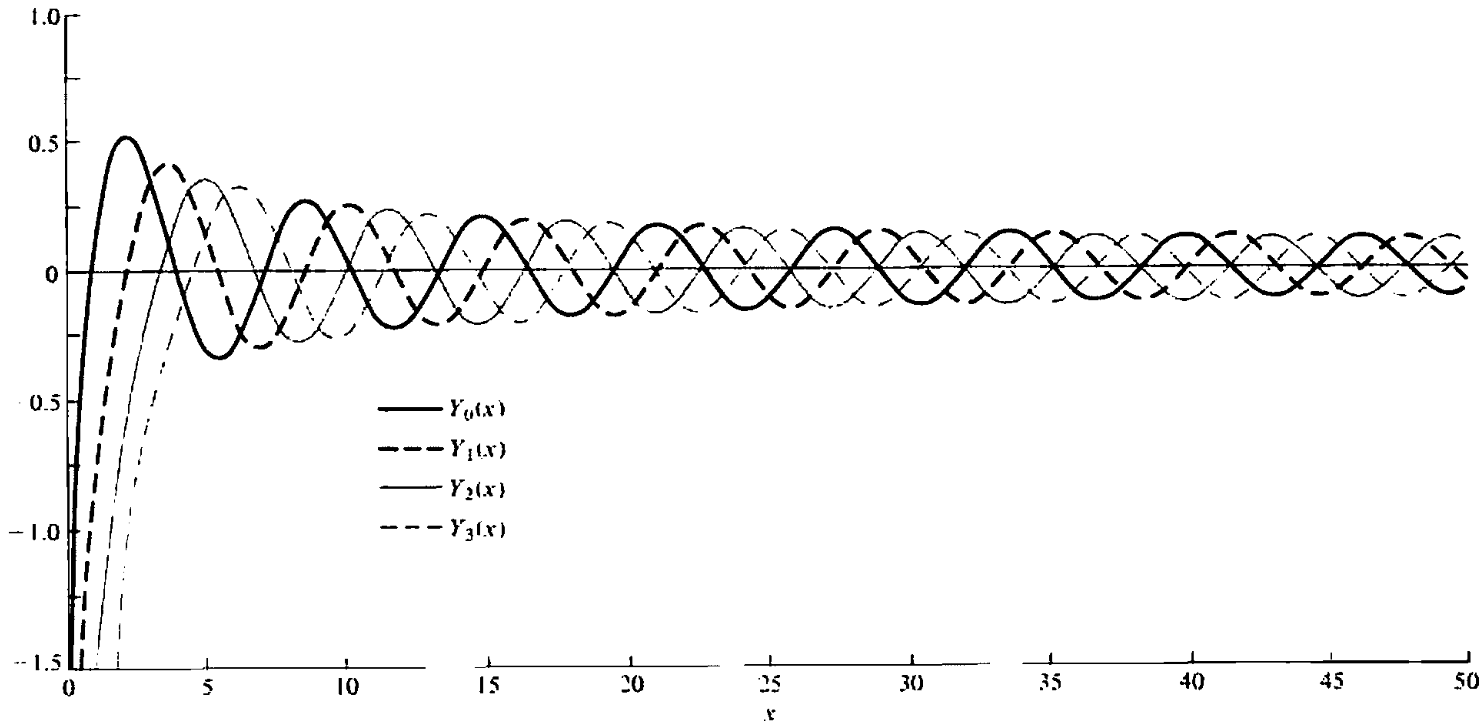


Figure V.2 Bessel functions of the second kind [$Y_0(x)$, $Y_1(x)$, $Y_2(x)$, $Y_3(x)$].

$J_1(x)/x$ FUNCTION

x	$J_1(x)/x$	x	$J_1(x)/x$	x	$J_1(x)/x$
0.0	0.50000	5.1	−0.06610	10.2	−0.00065
0.1	0.49938	5.2	−0.06600	10.3	−0.00304
0.2	0.49750	5.3	−0.06528	10.4	−0.00533
0.3	0.49440	5.4	−0.06395	10.5	−0.00751
0.4	0.49007	5.5	−0.06208	10.6	−0.00955
0.5	0.48454	5.6	−0.05970	10.7	−0.01144
0.6	0.47783	5.7	−0.05687	10.8	−0.01316
0.7	0.46999	5.8	−0.05363	10.9	−0.01471
0.8	0.46105	5.9	−0.05002	11.0	−0.01607
0.9	0.45105	6.0	−0.04611	11.1	−0.01724
1.0	0.44005	6.1	−0.04194	11.2	−0.01820
1.1	0.42809	6.2	−0.03757	11.3	−0.01896
1.2	0.41524	6.3	−0.03303	11.4	−0.01951
1.3	0.40156	6.4	−0.02838	11.5	−0.01986
1.4	0.38710	6.5	−0.02367	11.6	−0.02000
1.5	0.37196	6.6	−0.01894	11.7	−0.01994
1.6	0.35618	6.7	−0.01423	11.8	−0.01969
1.7	0.33986	6.8	−0.00959	11.9	−0.01924
1.8	0.32306	6.9	−0.00506	12.0	−0.01862
1.9	0.30587	7.0	−0.00067	12.1	−0.01783
2.0	0.28836	7.1	0.00354	12.2	−0.01688
2.1	0.27061	7.2	0.00755	12.3	−0.01579
2.2	0.25271	7.3	0.01131	12.4	−0.01457
2.3	0.23473	7.4	0.01481	12.5	−0.01324
2.4	0.21674	7.5	0.01803	12.6	−0.01180
2.5	0.19884	7.6	0.02095	12.7	−0.01029
2.6	0.18108	7.7	0.02355	12.8	−0.00871
2.7	0.16356	7.8	0.02582	12.9	−0.00707
2.8	0.14633	7.9	0.02774	13.0	−0.00541
2.9	0.12946	8.0	0.02933	13.1	−0.00373
3.0	0.11302	8.1	0.03057	13.2	−0.00205
3.1	0.09707	8.2	0.03146	13.3	−0.00039
3.2	0.08167	8.3	0.03202	13.4	0.00124
3.3	0.06687	8.4	0.03224	13.5	0.00282
3.4	0.05271	8.5	0.03213	13.6	0.00434
3.5	0.03925	8.6	0.03172	13.7	0.00578
3.6	0.02652	8.7	0.03100	13.8	0.00713
3.7	0.01455	8.8	0.03001	13.9	0.00838
3.8	0.00337	8.9	0.02875	14.0	0.00953
3.9	−0.00699	9.0	0.02726	14.1	0.01055
4.0	−0.01651	9.1	0.02554	14.2	0.01145
4.1	−0.02519	9.2	0.02363	14.3	0.01222
4.2	−0.03301	9.3	0.02155	14.4	0.01285
4.3	−0.03998	9.4	0.01932	14.5	0.01334
4.4	−0.04609	9.5	0.01697	14.6	0.01369
4.5	−0.05135	9.6	0.01453	14.7	0.01389
4.6	−0.05578	9.7	0.01202	14.8	0.01396
4.7	−0.05938	9.8	0.00947	14.9	0.01388
4.8	−0.06219	9.9	0.00691	15.0	0.01367
4.9	−0.06423	10.0	0.00435		
5.0	−0.06552	10.1	0.00182		

$\int_0^x J_0(\tau)\,d\tau$ AND $\int_0^x Y_0(\tau)\,d\tau$ FUNCTIONS

x	$\int_0^x J_0(\tau)\,d\tau$	$\int_0^x Y_0(\tau)\,d\tau$	x	$\int_0^x J_0(\tau)\,d\tau$	$\int_0^x Y_0(\tau)\,d\tau$
0.0	0.00000	0.00000	5.1	0.69920	0.16818
0.1	0.09991	−0.21743	5.2	0.68647	0.13551
0.2	0.19933	−0.34570	5.3	0.67716	0.10205
0.3	0.29775	−0.43928	5.4	0.67131	0.06814
0.4	0.39469	−0.50952	5.5	0.66891	0.03413
0.5	0.48968	−0.56179	5.6	0.66992	0.00035
0.6	0.58224	−0.59927	5.7	0.67427	−0.03284
0.7	0.67193	−0.62409	5.8	0.68187	−0.06517
0.8	0.75834	−0.63786	5.9	0.69257	−0.09630
0.9	0.84106	−0.64184	6.0	0.70622	−0.12595
1.0	0.91973	−0.63706	6.1	0.72263	−0.15385
1.1	0.99399	−0.62447	6.2	0.74160	−0.17975
1.2	1.06355	−0.60490	6.3	0.76290	−0.20344
1.3	1.12813	−0.57911	6.4	0.78628	−0.22470
1.4	1.18750	−0.54783	6.5	0.81147	−0.24338
1.5	1.24144	−0.51175	6.6	0.83820	−0.25931
1.6	1.28982	−0.47156	6.7	0.86618	−0.27239
1.7	1.33249	−0.42788	6.8	0.89512	−0.28252
1.8	1.36939	−0.38136	6.9	0.92470	−0.28966
1.9	1.40048	−0.33260	7.0	0.95464	−0.29377
2.0	1.42577	−0.28219	7.1	0.98462	−0.29486
2.1	1.44528	−0.23071	7.2	1.01435	−0.29295
2.2	1.45912	−0.17871	7.3	1.04354	−0.28811
2.3	1.46740	−0.12672	7.4	1.07190	−0.28043
2.4	1.47029	−0.07526	7.5	1.09917	−0.27002
2.5	1.46798	−0.02480	7.6	1.12508	−0.25702
2.6	1.46069	0.02420	7.7	1.14941	−0.24159
2.7	1.44871	0.07132	7.8	1.17192	−0.22392
2.8	1.43231	0.11617	7.9	1.19243	−0.20421
2.9	1.41181	0.15839	8.0	1.21074	−0.18269
3.0	1.38756	0.19765	8.1	1.22671	−0.15959
3.1	1.35992	0.23367	8.2	1.24021	−0.13516
3.2	1.32928	0.26620	8.3	1.25112	−0.10966
3.3	1.29602	0.29502	8.4	1.25939	−0.08335
3.4	1.26056	0.31996	8.5	1.26494	−0.05650
3.5	1.22330	0.34090	8.6	1.26777	−0.02940
3.6	1.18467	0.35775	8.7	1.26787	−0.00230
3.7	1.14509	0.37044	8.8	1.26528	0.02451
3.8	1.10496	0.37896	8.9	1.26005	0.05078
3.9	1.06471	0.38335	9.0	1.25226	0.07625
4.0	1.02473	0.38366	9.1	1.24202	0.10069
4.1	0.98541	0.38000	9.2	1.22946	0.12385
4.2	0.94712	0.37250	9.3	1.21473	0.14552
4.3	0.91021	0.36131	9.4	1.19799	0.16550
4.4	0.87502	0.34665	9.5	1.17944	0.18361
4.5	0.84186	0.32872	9.6	1.15927	0.19969
4.6	0.81100	0.30779	9.7	1.13772	0.21360
4.7	0.78271	0.28413	9.8	1.11499	0.22523
4.8	0.75721	0.25802	9.9	1.09134	0.23448
4.9	0.73468	0.22977	10.0	1.06701	0.24129
5.0	0.71531	0.19971			

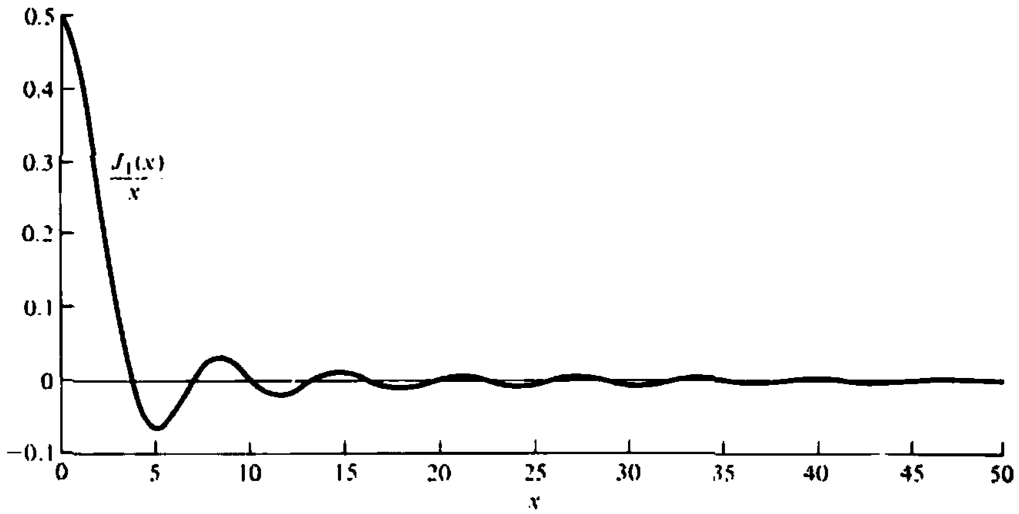


Figure V.3 Plot of $J_1(x)/x$ function.

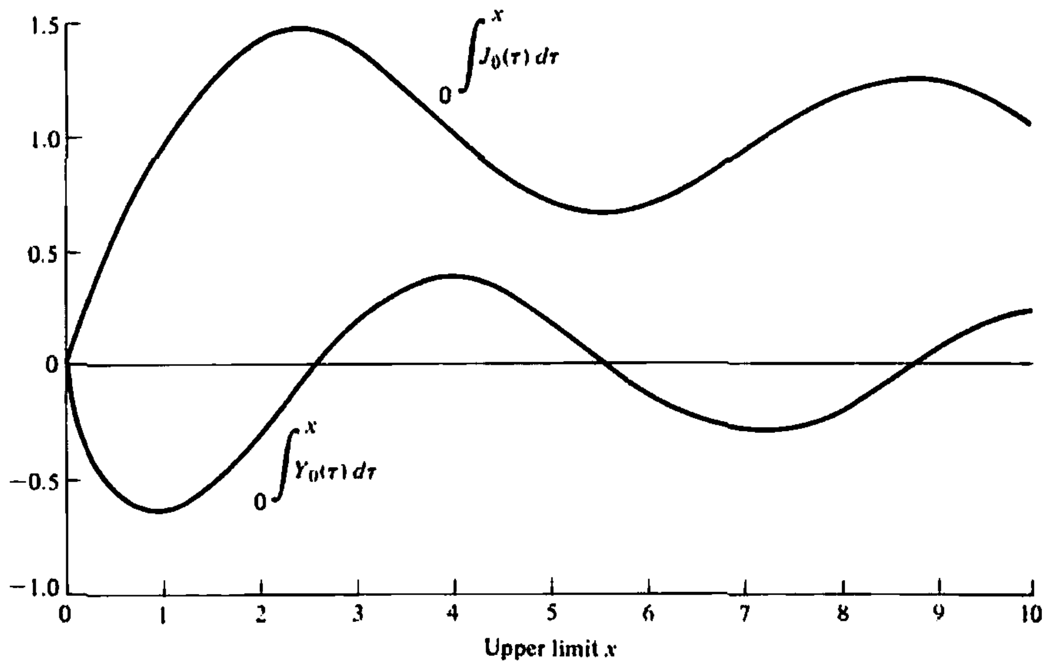


Figure V.4 Plots of $\int_0^x J_0(\tau)d\tau$ and $\int_0^x Y_0(\tau)d\tau$ functions.

APPENDIX VI

IDENTITIES

VI.1 TRIGONOMETRIC

1. Sum or difference:
   a. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
   b. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
   c. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
   d. $\cos(x - y) = \cos x \cos y + \sin x \sin y$
   e. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
   f. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
   g. $\sin^2 x + \cos^2 x = 1$
   h. $\tan^2 x - \sec^2 x = -1$
   i. $\cot^2 x - \csc^2 x = -1$
2. Sum or difference into products:
   a. $\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$
   b. $\sin x - \sin y = 2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$
   c. $\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$
   d. $\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$
3. Products into sum or difference:
   a. $2 \sin x \cos y = \sin(x + y) + \sin(x - y)$
   b. $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$
   c. $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$
   d. $2 \sin x \sin y = -\cos(x + y) + \cos(x - y)$
4. Double and half-angles:
   a. $\sin 2x = 2 \sin x \cos x$
   b. $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$
   c. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
   d. $\sin \dfrac{1}{2}x = \pm \sqrt{\dfrac{1 - \cos x}{2}}$ or $2 \sin^2 \theta = 1 - \cos 2\theta$

e. $\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ or $2\cos^2\theta = 1 + \cos 2\theta$

f. $\tan\frac{1}{2}x = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

5. Series:

a. $\sin x = \frac{e^{jx} - e^{-jx}}{2j} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

b. $\cos x = \frac{e^{jx} + e^{-jx}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

c. $\tan x = \frac{e^{jx} - e^{-jx}}{j(e^{jx} + e^{-jx})} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$

VI.2 HYPERBOLIC

1. Definitions:

a. Hyperbolic sine: $\sinh x = \frac{1}{2}(e^x - e^{-x})$

b. Hyperbolic cosine: $\cosh x = \frac{1}{2}(e^x + e^{-x})$

c. Hyperbolic tangent: $\tanh x = \frac{\sinh x}{\cosh x}$

d. Hyperbolic cotangent: $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$

e. Hyperbolic secant: $\operatorname{sech} x = \frac{1}{\cosh x}$

f. Hyperbolic cosecant: $\operatorname{csch} x = \frac{1}{\sinh x}$

2. Sum or difference:

a. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

b. $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

c. $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

d. $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

e. $\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

f. $\cosh^2 x - \sinh^2 x = 1$

g. $\tanh^2 x + \operatorname{sech}^2 x = 1$

h. $\coth^2 x - \operatorname{csch}^2 x = 1$

i. $\cosh(x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$

j. $\sinh(x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$

3. Series:

a. $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

b. $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

c. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

VI.3 LOGARITHMIC

1. $\log_b(MN) = \log_b M + \log_b N$
2. $\log_b(M/N) = \log_b M - \log_b N$
3. $\log_b(1/N) = -\log_b N$
4. $\log_b(M^n) = n \log_b M$
5. $\log_b(M^{1/n}) = \dfrac{1}{n} \log_b M$
6. $\log_a N = \log_b N \cdot \log_a b = \log_b N/\log_b a$
7. $\log_e N = \log_{10} N \cdot \log_e 10 = 2.302585 \log_{10} N$
8. $\log_{10} N = \log_e N \cdot \log_{10} e = 0.434294 \log_e N$

APPENDIX

VII

VECTOR ANALYSIS

VII.1 VECTOR TRANSFORMATIONS

In this appendix we present the vector transformations from rectangular-to-cylindrical (and vice-versa), from cylindrical-to-spherical (and vice-versa), and from rectangular-to-spherical (and vice-versa). The three coordinate systems are shown in Figure VII.1.

VII.1.1 Rectangular-to-Cylindrical (and Vice-Versa)

The coordinate transformation from rectangular (x, y, z) to cylindrical (ρ, ϕ, z) is given, referring to Figure VII.1(b)

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}\right\} \tag{VII-1}$$

In the rectangular coordinate system we express a vector $\mathbf{A}$ as

$$\mathbf{A} = \hat{\mathbf{a}}_x A_x + \hat{\mathbf{a}}_y A_y + \hat{\mathbf{a}}_z A_z \tag{VII-2}$$

where $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are the unit vectors and A_x, A_y, A_z are the components of the vector $\mathbf{A}$ in the rectangular coordinate system. We wish to write $\mathbf{A}$ as

$$\mathbf{A} = \hat{\mathbf{a}}_\rho A_\rho + \hat{\mathbf{a}}_\phi A_\phi + \hat{\mathbf{a}}_z A_z \tag{VII-3}$$

where $\hat{\mathbf{a}}_\rho$, $\hat{\mathbf{a}}_\phi$, $\hat{\mathbf{a}}_z$ are the unit vectors and A_ρ, A_ϕ, A_z are the vector components in the cylindrical coordinate system. The z-axis is common to both of them.

Referring to Figure VII.2, we can write

$$\left.\begin{aligned} \hat{\mathbf{a}}_x &= \hat{\mathbf{a}}_\rho\cos\phi - \hat{\mathbf{a}}_\phi\sin\phi \\ \hat{\mathbf{a}}_y &= \hat{\mathbf{a}}_\rho\sin\phi + \hat{\mathbf{a}}_\phi\cos\phi \\ \hat{\mathbf{a}}_z &= \hat{\mathbf{a}}_z \end{aligned}\right\} \tag{VII-4}$$

Using (VII-4) reduces (VII-2) to

$$\mathbf{A} = (\hat{\mathbf{a}}_\rho\cos\phi - \hat{\mathbf{a}}_\phi\sin\phi)A_x + (\hat{\mathbf{a}}_\rho\sin\phi + \hat{\mathbf{a}}_\phi\cos\phi)A_y + \hat{\mathbf{a}}_z A_z$$

$$\mathbf{A} = \hat{\mathbf{a}}_\rho(A_x\cos\phi + A_y\sin\phi) + \hat{\mathbf{a}}_\phi(-A_x\sin\phi + A_y\cos\phi) + \hat{\mathbf{a}}_z A_z \tag{VII-5}$$

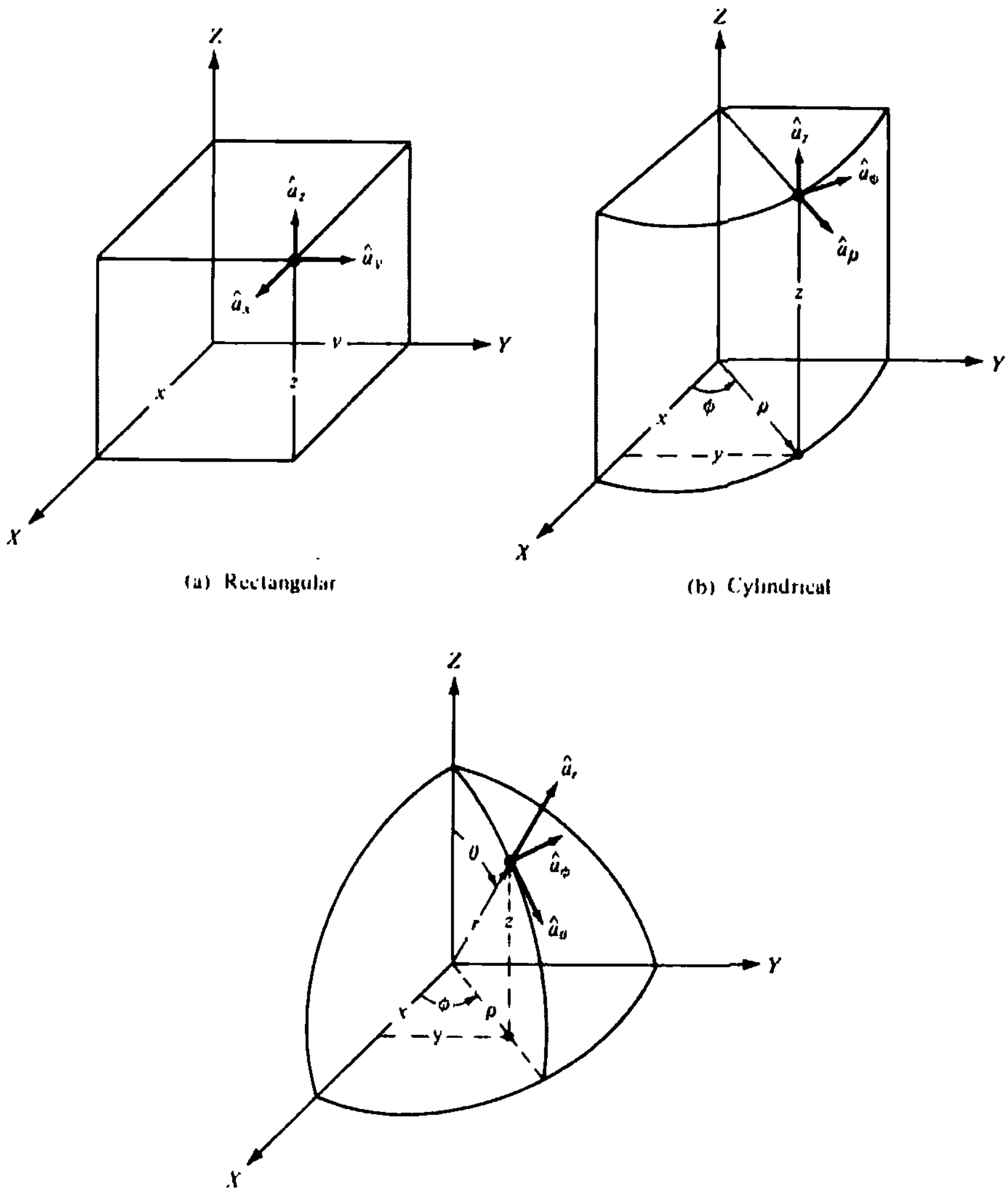


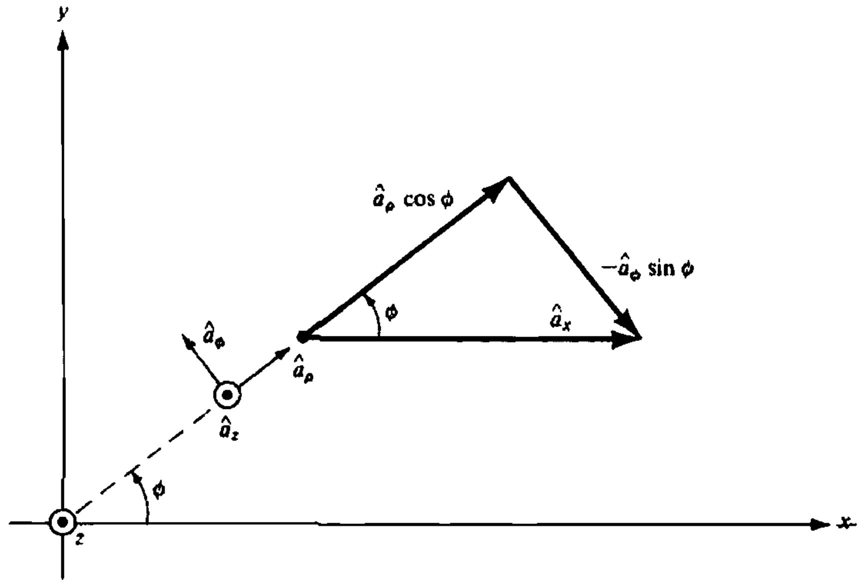
Figure VII.1 Rectangular, cylindrical, and spherical coordinate systems.

which when compared with (VII-3) leads to

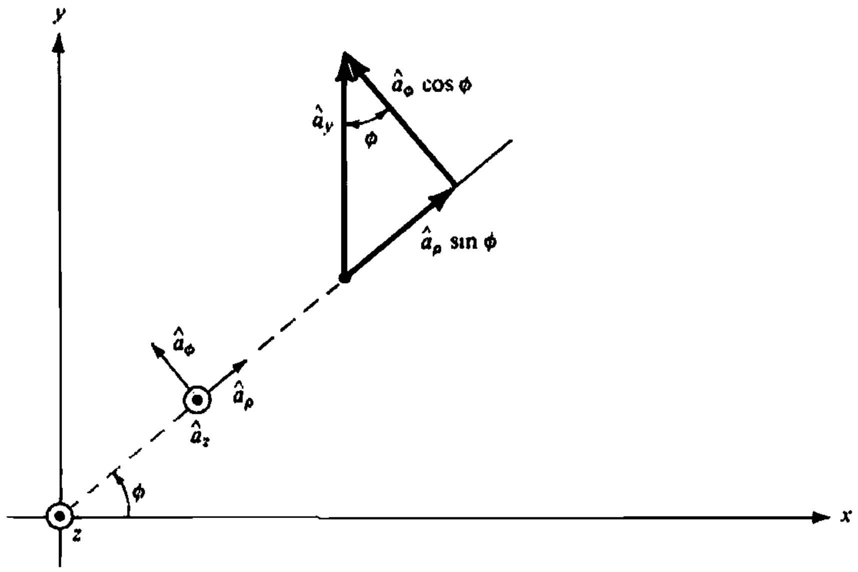
$$\left.\begin{aligned} A_\rho &= A_x \cos\phi + A_y \sin\phi \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \\ A_z &= A_z \end{aligned}\right\} \tag{VII-6}$$

In matrix form, (VII-6) can be written as

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{VII-6a}$$



(a) Geometry for unit vector $\hat{a}_x$



(b) Geometry for unit vector $\hat{a}_y$

Figure VII.2 Geometrical representation of transformations between unit vectors of rectangular and cylindrical coordinate systems.

where

$$[A]_{rc} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{VII-6b}$$

is the transformation matrix for rectangular-to-cylindrical components.

Since $[A]_{rc}$ is an orthonormal matrix (its inverse is equal to its transpose), we can write the transformation matrix for cylindrical-to-rectangular components as

$$[A]_{cr} = [A]_{rc}^{-1} = [A]_{rc}^{t} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{VII-7}$$

or

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} \tag{VII-7a}$$

or

$$\left.\begin{aligned} A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\ A_y &= A_\rho \sin\phi + A_\phi \cos\phi \\ A_z &= A_z \end{aligned}\right\} \tag{VII-7b}$$

VII.1.2 Cylindrical-to-Spherical (and Vice-Versa)

Referring to Figure VII.1(c), we can write that the cylindrical and spherical coordinates are related by

$$\left.\begin{aligned} \rho &= r\sin\theta \\ z &= r\cos\theta \end{aligned}\right\} \tag{VII-8}$$

In a geometrical approach similar to the one employed in the previous section, we can show that the cylindrical-to-spherical transformation of vector components is given by

$$\left.\begin{aligned} A_r &= A_\rho \sin\theta + A_z \cos\theta \\ A_\theta &= A_\rho \cos\theta - A_z \sin\theta \\ A_\phi &= A_\phi \end{aligned}\right\} \tag{VII-9}$$

or in matrix form by

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} \tag{VII-9a}$$

Thus the cylindrical-to-spherical transformation matrix can be written as

$$[A]_{cs} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \tag{VII-9b}$$

The $[A]_{cs}$ matrix is also orthonormal so that its inverse is given by

$$[A]_{sc} = [A]_{cs}^{-1} = [A]_{cs}^{t} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \tag{VII-10}$$

and the spherical-to-cylindrical transformation is accomplished by

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} \tag{VII-10a}$$

or

$$\left.\begin{aligned} A_\rho &= A_r \sin\theta + A_\theta \cos\theta \\ A_\phi &= A_\phi \\ A_z &= A_r \cos\theta - A_\theta \sin\theta \end{aligned}\right\} \qquad \text{(VII-10b)}$$

This time the component A_ϕ and coordinate ϕ are the same in both systems.

VII.1.3 Rectangular-to-Spherical (and Vice-Versa)

Many times it may be required that a transformation be performed directly from rectangular-to-spherical components. By referring to Figure VII.1, we can write that the rectangular and spherical coordinates are related by

$$\left.\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}\right\} \qquad \text{(VII-11)}$$

and the rectangular and spherical components by

$$\left.\begin{aligned} A_r &= A_x \sin\theta\cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta \\ A_\theta &= A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \end{aligned}\right\} \qquad \text{(VII-12)}$$

which can also be obtained by substituting (VII-6) into (VII-9). In matrix form, (VII-12) can be written as

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \qquad \text{(VII-12a)}$$

with the rectangular-to-spherical transformation matrix being

$$[A]_{rs} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \qquad \text{(VII-12b)}$$

The transformation matrix of (VII-12b) is also orthonormal so that its inverse can be written as

$$[A]_{sr} = [A]_{rs}^{-1} = [A]_{rs}^{t} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \qquad \text{(VII-13)}$$

and the spherical-to-rectangular components related by

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} \qquad \text{(VII-13a)}$$

or

$$\left.\begin{aligned} A_x &= A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi \\ A_y &= A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi \\ A_z &= A_r \cos\theta - A_\theta \sin\theta \end{aligned}\right\} \qquad \text{(VII-13b)}$$

VII.2 VECTOR DIFFERENTIAL OPERATORS

The differential operators of gradient of a scalar ($\nabla\psi$), divergence of a vector ($\nabla \cdot \mathbf{A}$), curl of a vector ($\nabla \times \mathbf{A}$), Laplacian of a scalar ($\nabla^2\psi$), and Laplacian of a vector ($\nabla^2\mathbf{A}$) frequently encountered in electromagnetic field analysis will be listed in the rectangular, cylindrical, and spherical coordinate systems.

VII.2.1 Rectangular Coordinates

$$\nabla\psi = \hat{\mathbf{a}}_x \frac{\partial\psi}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial\psi}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial\psi}{\partial z} \tag{VII-14}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{VII-15}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{a}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{a}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{a}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{VII-16}$$

$$\nabla \cdot \nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{VII-17}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{a}}_x \nabla^2 A_x + \hat{\mathbf{a}}_y \nabla^2 A_y + \hat{\mathbf{a}}_z \nabla^2 A_z \tag{VII-18}$$

VII.2.2 Cylindrical Coordinates

$$\nabla\psi = \hat{\mathbf{a}}_\rho \frac{\partial\psi}{\partial\rho} + \hat{\mathbf{a}}_\phi \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{a}}_z \frac{\partial\psi}{\partial z} \tag{VII-19}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{VII-20}$$

$$\begin{aligned} \nabla \times \mathbf{A} = {} & \hat{\mathbf{a}}_\rho \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\mathbf{a}}_\phi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) \\ & + \hat{\mathbf{a}}_z \left(\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right) \end{aligned} \tag{VII-21}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{VII-22}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \tag{VII-23}$$

or in an expanded form

$$\begin{aligned} \nabla^2\mathbf{A} = {} & \hat{\mathbf{a}}_\rho \left(\frac{\partial^2 A_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\rho}{\partial\rho} - \frac{A_\rho}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial\phi^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial^2 A_\rho}{\partial z^2}\right) \\ & + \hat{\mathbf{a}}_\phi \left(\frac{\partial^2 A_\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\rho} - \frac{A_\phi}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} + \frac{\partial^2 A_\phi}{\partial z^2}\right) \\ & + \hat{\mathbf{a}}_z \left(\frac{\partial^2 A_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \end{aligned} \tag{VII-23a}$$

In the cylindrical coordinate system $\nabla^2\mathbf{A} \neq \hat{\mathbf{a}}_\rho \nabla^2 A_\rho + \hat{\mathbf{a}}_\phi \nabla^2 A_\phi + \hat{\mathbf{a}}_z \nabla^2 A_z$ because the orientation of the unit vectors $\hat{\mathbf{a}}_\rho$ and $\hat{\mathbf{a}}_\phi$ varies with the ρ and ϕ coordinates.

VII.2.3 Spherical Coordinates

$$\nabla\psi = \hat{\mathbf{a}}_r \frac{\partial\psi}{\partial r} + \hat{\mathbf{a}}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\mathbf{a}}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{VII-24}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{VII-25}$$

$$\begin{aligned}\nabla\times\mathbf{A} = {} & \frac{\hat{\mathbf{a}}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\mathbf{a}}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] \\ & + \frac{\hat{\mathbf{a}}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned} \tag{VII-26}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{VII-27}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A} \tag{VII-28}$$

or in an expanded form

$$\begin{aligned}\nabla^2\mathbf{A} = {} & \hat{\mathbf{a}}_r\left(\frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}A_r + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right. \\ & \left. - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\ & + \hat{\mathbf{a}}_\theta\left(\frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\theta}{\partial\theta}\right. \\ & \left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\ & + \hat{\mathbf{a}}_\phi\left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} - \frac{1}{r^2\sin^2\theta}A_\phi + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\phi}{\partial\theta}\right. \\ & \left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\end{aligned} \tag{VII-28a}$$

Again note that $\nabla^2\mathbf{A} \neq \hat{\mathbf{a}}_r\nabla^2 A_r + \hat{\mathbf{a}}_\theta\nabla^2 A_\theta + \hat{\mathbf{a}}_\phi\nabla^2 A_\phi$ since the orientation of the unit vectors $\hat{\mathbf{a}}_r$, $\hat{\mathbf{a}}_\theta$, and $\hat{\mathbf{a}}_\phi$ varies with the r, θ, and ϕ coordinates.

VII.3 VECTOR IDENTITIES

VII.3.1 Addition and Multiplication

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 \tag{VII-29}$$

$$\mathbf{A}\cdot\mathbf{A}^* = |\mathbf{A}|^2 \tag{VII-30}$$

$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A} \tag{VII-31}$$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \tag{VII-32}$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{VII-33}$$

$$(\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C} \tag{VII-34}$$

$$(\mathbf{A}+\mathbf{B})\times\mathbf{C} = \mathbf{A}\times\mathbf{C} + \mathbf{B}\times\mathbf{C} \tag{VII-35}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} \tag{VII-36}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{VII-37}$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \mathbf{A}\cdot\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\\ &= \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{D}\mathbf{C} - \mathbf{B}\cdot\mathbf{C}\mathbf{D})\\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\end{aligned} \tag{VII-38}$$

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D} \tag{VII-39}$$

VII.3.2 Differentiation

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{VII-40}$$

$$\nabla\times\nabla\psi = 0 \tag{VII-41}$$

$$\nabla(\phi+\psi) = \nabla\phi + \nabla\psi \tag{VII-42}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{VII-43}$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \tag{VII-44}$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \tag{VII-45}$$

$$\nabla\cdot(\psi\mathbf{A}) = \mathbf{A}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{A} \tag{VII-46}$$

$$\nabla\times(\psi\mathbf{A}) = \nabla\psi\times\mathbf{A} + \psi\nabla\times\mathbf{A} \tag{VII-47}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \tag{VII-48}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B} \tag{VII-49}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{VII-50}$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{VII-51}$$

VII.3.3 Integration

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} \qquad \text{Stoke's theorem} \tag{VII-52}$$

$$\oiint_S \mathbf{A}\cdot d\mathbf{s} = \iiint_V (\nabla\cdot\mathbf{A})\,dv \qquad \text{Divergence theorem} \tag{VII-53}$$

$$\oiint_S (\hat{\mathbf{n}}\times\mathbf{A})\,ds = \iiint_V (\nabla\times\mathbf{A})\,dv \tag{VII-54}$$

$$\oiint_S \psi\,d\mathbf{s} = \iiint_V \nabla\psi\,dv \tag{VII-55}$$

$$\oint_C \psi\,d\mathbf{l} = \iint_S \hat{\mathbf{n}}\times\nabla\psi\,ds \tag{VII-56}$$

APPENDIX

VIII

METHOD OF STATIONARY PHASE

In many problems, the following integral is often encountered and in most cases cannot be integrated exactly:

$$I(k) = \int_a^b \int_c^d F(x, y) e^{jkf(x,y)}\, dx\, dy \tag{VIII-1}$$

where

k = real

$f(x, y)$ = real, independent of k, and nonsingular

$F(x, y)$ = may be complex, independent of k, and nonsingular

Often, however, the above integral needs to be evaluated only for large values of k, but the task is still formidable. An approximate technique, known as the *Method of Stationary Phase*, exists that can be used to obtain an approximate asymptotic expression, for large values of k, for the above integral.

The method is justified by the asymptotic approximation of the single integral

$$I'(k) = \int_a^b F(x) e^{jkf(x)}\, dx \tag{VIII-2}$$

where

k = real

$f(x, y)$ = real, independent of k, and nonsingular

$F(x, y)$ = may be complex, independent of k, and nonsingular

which can be extended to include double integrals.

The asymptotic evaluation of (VIII-1) for large k is based on the following: $f(x, y)$ is a well behaved function and its variation near the stationary points x_s, y_s determined by

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_s \\ y=y_s}} \equiv f'_x(x_s, y_s) = 0 \tag{VIII-3a}$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_s\\y=y_s}} \equiv f'_y(x_s, y_s) = 0 \tag{VIII-3b}$$

is slow varying. Outside these regions, the function $f(x, y)$ varies faster such that the exponential factor $\exp[jkf(x, y)]$ of the integrand oscillates very rapidly between the values of $+1$ and -1, for large values of k. Assuming $F(x, y)$ is everywhere a slow varying function, the contributions to the integral outside the stationary points tend to cancel each other. Thus the only contributors to the integral, in an approximate sense, are the stationary points and their neighborhoods. Thus, we can write (VIII-1) approximately as

$$\begin{aligned} I(k) &\simeq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x_s, y_s)e^{jkf(x,y)}\,dx\,dy \\ &= F(x_s, y_s)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{jkf(x,y)}\,dx\,dy \end{aligned} \tag{VIII-4}$$

where the limits have been extended, for convenience, to infinity since the net contribution outside the stationary points and their near regions is negligible.

In the neighborhood of the stationary points, the function $f(x, y)$ can be approximated by a truncated Taylor series

$$\begin{aligned} f(x, y) \simeq f(x_s, y_s) &+ \frac{1}{2}(x - x_s)^2 f''_{xx}(x_s, y_s) + \frac{1}{2}(y - y_s)^2 f''_{yy}(x_s, y_s) \\ &+ (x - x_s)(y - y_s)f''_{xy}(x_s, y_s) \end{aligned} \tag{VIII-5}$$

since

$$f'_x(x_s, y_s) = f'_y(x_s, y_s) = 0 \tag{VIII-6}$$

by (VIII-3a) and (VIII-3b). For convenience, we have adopted the notation

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=x_s\\y=y_s}} \equiv f''_{xx}(x_s, y_s) \tag{VIII-7a}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\substack{x=x_s\\y=y_s}} \equiv f''_{yy}(x_s, y_s) \tag{VIII-7b}$$

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{\substack{x=x_s\\y=y_s}} \equiv f''_{xy}(x_s, y_s) \tag{VIII-7c}$$

For brevity, we write (VIII-5) as

$$f(x, y) \simeq f(x_s, y_s) + A\xi^2 + B\eta^2 + C\xi\eta \tag{VIII-8}$$

where

$$A = \frac{1}{2}f''_{xx}(x_s, y_s) \tag{VIII-8a}$$

$$B = \frac{1}{2}f''_{yy}(x_s, y_s) \tag{VIII-8b}$$

$$C = f''_{xy}(x_s, y_s) \tag{VIII-8c}$$

$$\xi = (x - x_s) \tag{VIII-8d}$$

$$\eta = (y - y_s) \tag{VIII-8e}$$

Using (VIII-8)–(VIII-8e) reduces (VIII-4) to

$$I(k) = F(x_s, y_s)e^{jkf(x_s,y_s)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{jk(A\xi^2 + B\eta^2 + C\xi\eta)}\, d\xi\, d\eta \tag{VIII-9}$$

We now write the quadratic factor of the exponential, by a proper rotation of the coordinate axes ξ, η to μ, λ, in a diagonal form as

$$A\xi^2 + B\eta^2 + C\xi\eta = A'\mu^2 + B'\lambda^2 \tag{VIII-10}$$

related to A, B, and C by

$$A' = \frac{1}{2}[(A + B) + \sqrt{(A + B)^2 - (4AB - C^2)}] \tag{VIII-10a}$$

$$B' = \frac{1}{2}[(A + B) - \sqrt{(A + B)^2 - (4AB - C^2)}] \tag{VIII-10b}$$

which are found by solving the secular determinant

$$\begin{vmatrix} (A - \zeta) & C/2 \\ C/2 & (B - \zeta) \end{vmatrix} = 0 \tag{VIII-11}$$

with $\zeta_1 = A'$ and $\zeta_2 = B'$. Substituting (VIII-10) into (VIII-9) we can write

$$I(k) = F(x_s, y_s)e^{jkf(x_s,y_s)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{jk(A'\mu^2 + B'\lambda^2)}\, d\mu\, d\lambda$$

$$I(k) = F(x_s, y_s)e^{jkf(x_s,y_s)} \int_{-\infty}^{+\infty} e^{\pm jk|A'|\mu^2}\, d\mu \int_{-\infty}^{+\infty} e^{\pm jk|B'|\lambda^2}\, d\lambda \tag{VIII-12}$$

where the signs in the exponents are determined by the signs of A' and B', which in turn depend upon A and B, as given in (VIII-10a) and (VIII-10b). The two integrals in (VIII-12) are of the same form and can be evaluated by examining the integral

$$I''(k) = \int_{-\infty}^{+\infty} e^{\pm jk|\alpha|t^2}\, dt = 2\int_0^{\infty} e^{\pm jk|\alpha|t^2}\, dt \tag{VIII-13}$$

where α can represent either A' or B' of (VIII-12). Making a change of variable by letting

$$k|\alpha|t^2 = \frac{\pi}{2}\tau^2 \tag{VIII-13a}$$

$$dt = \sqrt{\frac{\pi}{2k|\alpha|}}\, d\tau \tag{VIII-13b}$$

we can rewrite (VIII-13) as

$$I''(k) = 2\sqrt{\frac{\pi}{2k|\alpha|}} \int_0^{\infty} e^{\pm j\frac{\pi}{2}\tau^2}\, d\tau \tag{VIII-14}$$

The integral is recognized as being the complex Fresnel integral whose value is

$$\int_0^{\infty} e^{\pm j\frac{\pi}{2}\tau^2}\, d\tau = \frac{1}{2}(1 \pm j) = \frac{1}{\sqrt{2}} e^{\pm j\frac{\pi}{4}} \tag{VIII-15}$$

which can be used to write (VIII-14) as

$$I''(k) = 2\sqrt{\frac{\pi}{2k|\alpha|}}\int_0^{\infty} e^{\pm j\frac{\pi}{2}\tau^2}\,d\tau = \sqrt{\frac{\pi}{k|\alpha|}}\,e^{\pm j\frac{\pi}{4}} \tag{VIII-16}$$

The result of (VIII-16) can be used to reduce (VIII-12) to

$$I(k) \simeq F(x_s, y_s)\,e^{jkf(x_s,y_s)}\,\frac{\pi}{k\sqrt{|A'||B'|}}\,e^{\pm j\frac{\pi}{4}}\,e^{\pm j\frac{\pi}{4}} \tag{VIII-17}$$

If A' and B' are both positive, then $e^{+j\frac{\pi}{4}}e^{+j\frac{\pi}{4}} = e^{+j\frac{\pi}{2}} = +j$

If A' and B' are both negative, then $e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$

If A' and B' have different signs, then $e^{\pm j\frac{\pi}{4}}e^{\mp j\frac{\pi}{4}} = 1$

Thus, (VIII-17) can be cast into the form

$$I(k) \simeq F(x_s, y_s)\,e^{jkf(x_s,y_s)}\,\frac{j\pi\delta}{k\sqrt{|A'||B'|}} \tag{VIII-18}$$

where

$$\delta = \begin{cases} +1 & \text{if } A' \text{ and } B' \text{ are both positive} \\ -1 & \text{if } A' \text{ and } B' \text{ are both negative} \\ -j & \text{if } A' \text{ and } B' \text{ have different signs} \end{cases} \tag{VIII-18a}$$

Examining (VIII-10a) and (VIII-10b), it is clear that

(a) A' and B' are real (because A, B, and C are real)
(b) $A' + B' = A + B$ (VIII-19)
(c) $A'B' = (4AB - C^2)/4$

Using the results of (VIII-19), we reduce (VIII-18) to

$$I(k) \simeq F(x_s, y_s)\,e^{jkf(x_s,y_s)}\,\frac{j2\pi\delta}{k\sqrt{|4AB - C^2|}}. \tag{VIII-20}$$

To determine the signs of A' and B', let us refer to (VIII-19).

(a) If $4AB > C^2$, then A and B have the same sign and $A'B' > 0$. Thus, A' and B' have the same sign.
   1. If $A > 0$ then $B > 0$ and $A' > 0$, $B' > 0$
   2. If $A < 0$ then $B < 0$ and $A' < 0$, $B' < 0$

(b) If $4AB < C^2$, then $A'B' < 0$, and A' and B' have different signs. Summarizing the results we can write that
   1. If $4AB > C^2$ and $A > 0$, then A' and B' are both positive
   2. If $4AB > C^2$ and $A < 0$, then A' and B' are both negative
   3. If $4AB < C^2$, then A' and B' have different signs

Using the preceding deductions, we can write the sign information of (VIII-18a) as

$$\delta = \begin{cases} +1 & \text{if } 4AB > C^2 \text{ and } A > 0 \\ -1 & \text{if } 4AB > C^2 \text{ and } A < 0 \\ -j & \text{if } 4AB < C^2 \end{cases} \tag{VIII-21}$$

in the evaluation of the integral in

$$I(k) \simeq F(x_s, y_s)\, e^{jkf(x_s, y_s)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{jk(A\xi^2 + B\eta^2 + C\xi\eta)}\, d\xi\, d\eta$$

$$I(k) \simeq F(x_s, y_s)\, e^{jkf(x_s, y_s)} \frac{j2\pi\delta}{k\sqrt{|4AB - C^2|}} \tag{VIII-22}$$

APPENDIX

IX

TELEVISION, RADIO, TELEPHONE, AND RADAR FREQUENCY SPECTRUMS

IX.1 TELEVISION

IX.1.1 *V*ery *H*igh *F*requency (VHF) Channels

CHANNEL NUMBER	2 3 4	5 6	7 8 9 10 11 12 13
FREQUENCY (MHz)	54↑60↑66↑72	76↑82↑88	174↑180↑186↑192↑198↑204↑210↑216

IX.1.2 *U*ltra *H*igh *F*requency (UHF) Channels*

CHANNEL NUMBER	14 15 16 17 18 19 20 . . . 82 83
FREQUENCY (MHz)	470↑476↑482↑488↑494↑500↑506↑512 . . . 878↑884↑890

For both VHF and UHF channels, each channel has a 6-MHz bandwidth. For each channel, the carrier frequency for the video part is equal to the lower frequency of the bandwidth plus 1.25 MHz while the carrier frequency for the audio part is equal to the upper frequency of the bandwidth minus 0.25 MHz.

*In top ten urban areas in the United States, land mobile is allowed in the first seven UHF TV channels (470–512 MHz).

Examples: Channel 2 (VHF): f_0 (video) = 54 + 1.25 = 55.25 MHz
f_0 (audio) = 60 − 0.25 = 59.75 MHz

Channel 14 (UHF): f_0 (video) = 470 + 1.25 = 471.25 MHz
f_0 (audio) = 476 − 0.25 = 475.75 MHz

IX.2 RADIO

IX.2.1 *A*mplitude *M*odulation (AM) Radio

Number of channels: 107 (each with 10-kHz separation)
Frequency range: 535–1605 kHz

IX.2.2 *F*requency *M*odulation (FM) Radio

Number of channels: 100 (each with 200-kHz separation)
Frequency range: 88–108 MHz

IX.3 AMATEUR BANDS

Band	Frequency (MHz)	Band	Frequency (MHz)
160-m	1.8–2.0	2-m	144.0–148.0
80-m	3.5–4.0	—	220–225
40-m	7.0–7.3	—	420–450
20-m	14.0–14.35	—	1215–1300
15-m	21.0–21.45	—	2300–2450
10-m	28.0–29.7	—	3300–3500
6-m	50.0–54.0	—	5650–5925

IX.4 CELLULAR TELEPHONE

IX.4.1 Land Mobile Systems

Uplink: MS to BS (mobile station to base station)

Downlink: BS to MS (base station to mobile station)

	UPLINK (MHz)	DOWNLINK (MHz)
United States of America (IS-54):	869–894	824–849
Europe–Asia (GSM): Global System for Mobile communications	890–915	935–960
Japan (NTT): Nippon Telegraph & Telephone Corporation	870–885	925–940

IX.4.2 Cordless Telephone

United States of America: 46–49 MHz
Digital European Cordless Telecommunications (DECT): 1.880–1.990 GHz

IX.5 RADAR IEEE BAND DESIGNATIONS

HF (High Frequency):	3–30	MHz
VHF (Very High Frequency):	30–300	MHz
UHF (Ultra High Frequency):	300–1,000	MHz
L-band:	1–2	GHz
S-band:	2–4	GHz
C-band:	4–8	GHz
X-band:	8–12	GHz
K_u-band:	12–18	GHz
K-band:	18–27	GHz
K_a-band:	27–40	GHz
Millimeter wave band:	40–300	GHz