5. Expected Utility Theory

5.1 Money lotteries and attitudes to risk

The introduction of chance moves gives rise to probabilistic outcomes, which we called lotteries. In Chapter 4 we restricted attention to lotteries whose outcomes are sums of money (money lotteries) and to one possible way of ranking such lotteries, based on the notion of risk neutrality. In this section we will continue to focus on money lotteries and define other possible attitudes to risk.[1]

As before, we restrict attention to finite lotteries. Recall that a money lottery is a probability distribution of the form

$$\begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$

(with $0 \leq p_i \leq 1$, for all $i = 1, 2, \dots, n$, and $p_1 + p_2 + \cdots + p_n = 1$) and that (Definition 4.2.2, Chapter 4) its expected value is the number $(x_1 p_1 + x_2 p_2 + \cdots + x_n p_n)$.
If L is a money lottery, we denote by $\mathbb{E}[L]$ the expected value of L. Thus, for example, if

$$L = \begin{pmatrix} \$30 & \$45 & \$90 \\ \frac{1}{3} & \frac{5}{9} & \frac{1}{9} \end{pmatrix} \quad \text{then} \quad \mathbb{E}[L] = \tfrac{1}{3}(30) + \tfrac{5}{9}(45) + \tfrac{1}{9}(90) = 45.$$

Recall also (Definition 4.2.3, Chapter 4) that a person is said to be *risk neutral* if she considers a money lottery to be just as good as its expected value for certain. For example, a risk-neutral person would consider getting \$45 with certainty to be just as good as playing lottery $L = \begin{pmatrix} \$30 & \$45 & \$90 \\ \frac{1}{3} & \frac{5}{9} & \frac{1}{9} \end{pmatrix}$.
We can now consider different attitudes to risk, besides risk neutrality.

[1] In the next section we will consider more general lotteries, where the outcomes need not be sums of money, and introduce the theory of expected utility.

Definition 5.1.1 Let L be a money lottery and consider the choice between L and getting $\$\mathbb{E}[L]$ (the expected value of L) for certain . Then

- An individual who prefers $\$\mathbb{E}[L]$ for certain to L is said to be *risk averse*.
- An individual who is indifferent between $\$\mathbb{E}[L]$ for certain and L is said to be *risk neutral*.
- An individual who prefers L to $\$\mathbb{E}[L]$ for certain is said to be *risk loving*.

Note that if an individual is risk **neutral**, has transitive preferences over money lotteries and prefers more money to less, then we can tell how that individual ranks any two money lotteries. For example, how would a risk neutral individual rank the two lotteries $L_1 = \begin{pmatrix} \$30 & \$45 & \$90 \\ \frac{1}{3} & \frac{5}{9} & \frac{1}{9} \end{pmatrix}$ and $L_2 = \begin{pmatrix} \$5 & \$100 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$?

Since $\mathbb{E}[L_1] = 45$ and the individual is risk neutral, $L_1 \sim \$45$; since $\mathbb{E}[L_2] = 43$ and the individual is risk neutral, $\$43 \sim L_2$; since the individual prefers more money to less, $\$45 \succ \$43$; thus, by transitivity, $L_1 \succ L_2$.

On the other hand, knowing that an individual is risk **averse**, has transitive preferences over money lotteries and prefers more money to less is not sufficient to predict how she will choose between two arbitrary money lotteries.
For example, as we will see later (see Exercise 5.11), it is possible that one risk-averse individual will prefer $L_3 = \begin{pmatrix} \$28 \\ 1 \end{pmatrix}$ (whose expected value is 28) to $L_4 = \begin{pmatrix} \$10 & \$50 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (whose expected value is 30), while another risk-averse individual will prefer L_4 to L_3.

Similarly, knowing that an individual is risk **loving**, has transitive preferences over money lotteries and prefers more money to less is not sufficient to predict how she will choose between two arbitrary money lotteries.

R Note that "rationality" does not, and should not, dictate whether an individual should be risk neutral, risk averse or risk loving: an individual's attitude to risk is merely a reflection of that individual's preferences. It is a generally accepted principle that *de gustibus non est disputandum* (in matters of taste, there can be no disputes). According to this principle, there is no such thing as an irrational preference and thus there is no such thing as an irrational attitude to risk. From an empirical point of view, however, most people reveal through their choices (e.g. the decision to buy insurance) that they are risk averse, at least when the stakes are high.

As noted above, with the exception of risk-neutral individuals, even if we restrict attention to money lotteries we are not able to say much – in general – about how an individual would choose among lotteries. What we need is a theory of "rational" preferences over lotteries that (1) is general enough to cover lotteries whose outcomes are not necessarily sums of money and (2) is capable of accounting for different attitudes to risk in the case of money lotteries. One such theory is the theory of expected utility, to which we now turn.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 5.4.1 at the end of this chapter.

5.2 Expected utility: theorems

The theory of expected utility was developed by the founders of game theory, namely John von Neumann and Oskar Morgenstern, in their 1944 book *Theory of Games and Economic Behavior*. In a rather unconventional way, we shall first (in this section) state the main result of the theory (which we split into two theorems) and then (in the following section) explain the assumptions (or axioms) behind that result. The reader who is not interested in understanding the conceptual foundations of expected utility theory, but wants to understand what the theory says and how it can be used, can study this section and skip the next.

Let O be a set of *basic outcomes*. Note that a basic outcome need not be a sum of money: it could be the state of an individual's health, or whether the individual under consideration receives an award, or whether it will rain on the day of her planned outdoor party, etc.

Let $\mathscr{L}(O)$ be the set of *simple lotteries* (or probability distributions) over O. We will assume throughout that O is a finite set: $O = \{o_1, o_2, ..., o_m\}$ $(m \geq 1)$.

Thus, an element of $\mathscr{L}(O)$ is of the form $\begin{pmatrix} o_1 & o_2 & \dots & o_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$ with $0 \leq p_i \leq 1$, for all $i = 1,2,...,m$, and $p_1 + p_2 + ... + p_m = 1$.

We will use the symbol L (with or without subscript) to denote an element of $\mathscr{L}(O)$, that is, a simple lottery. Lotteries are used to represent situations of uncertainty. For example, if $m = 4$ and the individual faces the lottery $L = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{2}{5} & 0 & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$ then she knows that, eventually, the outcome will be one and only one of o_1, o_2, o_3, o_4, but does not know which one; furthermore, she is able to quantify her uncertainty by assigning probabilities to these outcomes.

We interpret these probabilities either as objectively obtained from relevant (past) data or as subjective estimates by the individual. For example, an individual who is considering whether or not to insure her bicycle against theft for the following 12 months knows that there are two relevant basic outcomes: either the bicycle will be stolen or it will not be stolen. Furthermore, she can look up data on past bicycle thefts in her area and use the proportion of bicycles that were stolen as an "objective" estimate of the probability that her bicycle will be stolen. Alternatively, she can use a more subjective estimate: for example she might use a lower probability of theft than suggested by the data because she knows herself to be very conscientious and – unlike other people – to always lock her bicycle when left unattended.

The assignment of *zero probability* to a particular basic outcome is taken to be an expression of *belief, not impossibility*: the individual is confident that the outcome will not arise, but she cannot rule out that outcome on logical grounds or by appealing to the laws of nature.

Among the elements of $\mathscr{L}(O)$ there are the degenerate lotteries that assign probability 1 to one basic outcome: for example, if $m = 4$ one degenerate lottery is $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

To simplify the notation we will often denote degenerate lotteries as basic outcomes, that is, instead of writing $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ we will simply write o_3.

Thus, in general, the degenerate lottery $\begin{pmatrix} o_1 & \dots & o_{i-1} & o_i & o_{i+1} & \dots & o_m \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ will be denoted by o_i. As another simplification, we will often omit those outcomes that are assigned zero probability. For example, if $m = 4$, the lottery $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{pmatrix}$ will be written more simply as $\begin{pmatrix} o_1 & o_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

In this chapter we shall call the individual under consideration the Decision-Maker, or *DM* for short. The theory of expected utility assumes that the *DM* has a complete and transitive ranking $\succsim$ of the elements of $\mathscr{L}(O)$ (indeed, this is one of the axioms listed in the next section). As in Chapter 2, the interpretation of $L \succsim L'$ is that the *DM* considers L to be at least as good as L'. By completeness, given any two lotteries L and L', either $L \succ L'$ (the *DM* prefers L to L') or $L' \succ L$ (the *DM* prefers L' to L) or $L \sim L'$ (the *DM* is indifferent between L and L'). Furthermore, by transitivity, for any three lotteries L_1, L_2 and L_3, if $L_1 \succsim L_2$ and $L_2 \succsim L_3$, then $L_1 \succsim L_3$. Besides completeness and transitivity, a number of other "rationality" constraints are postulated on the ranking $\succsim$ of the elements of $\mathscr{L}(O)$; these constraints are the so-called Expected Utility Axioms and are discussed in the next section.

Definition 5.2.1 A ranking $\succsim$ of the elements of $\mathscr{L}(O)$ that satisfies the Expected Utility Axioms (listed in the next section) is called a *von Neumann-Morgenstern ranking*.

The following two theorems are the key results in the theory of expected utility.

Theorem 5.2.1 [von Neumann-Morgenstern, 1944]. Let $O = \{o_1, o_2, ..., o_m\}$ be a set of basic outcomes and let $\mathscr{L}(O)$ be the set of simple lotteries over O. If $\succsim$ is a von Neumann-Morgenstern ranking of the elements of $\mathscr{L}(O)$ then there exists a function $U : O \to \mathbb{R}$, called a *von Neumann-Morgenstern utility function*,indexvon Neumann-Morgenstern!utility function that assigns a number (called *utility*) to every basic outcome and is such that, for any two lotteries $L = \begin{pmatrix} o_1 & o_2 & \dots & o_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$ and $L' = \begin{pmatrix} o_1 & o_2 & \dots & o_m \\ q_1 & q_2 & \dots & q_m \end{pmatrix}$,

$$L \succ L' \text{ if and only if } \mathbb{E}[U(L)] > \mathbb{E}[U(L')], \text{ and}$$
$$L \sim L' \text{ if and only if } \mathbb{E}[U(L)] = \mathbb{E}[U(L')], \text{ where}$$
$$U(L) = \begin{pmatrix} U(o_1) & U(o_2) & \dots & U(o_m) \\ p_1 & p_2 & \dots & p_m \end{pmatrix}, \; U(L') = \begin{pmatrix} U(o_1) & U(o_2) & \dots & U(o_m) \\ q_1 & q_2 & \dots & q_m \end{pmatrix},$$

$\mathbb{E}[U(L)]$ is the expected value of the lottery $U(L)$ and $\mathbb{E}[U(L')]$ is the expected value of the lottery $U(L')$, that is,

$$\mathbb{E}[U(L)] = p_1 U(o_1) + p_2 U(o_2) + ... + p_m U(o_m), \text{ and}$$
$$\mathbb{E}[U(L')] = q_1 U(o_1) + q_2 U(o_2) + ... + q_m U(o_m).$$

$\mathbb{E}[U(L)]$ is called the *expected utility* of lottery L (and $\mathbb{E}[U(L')]$ the expected utility of lottery L'). We say that any function $U : O \to \mathbb{R}$ that satisfies the property that, for any two lotteries L and L', $L \succsim L'$ if and only if $\mathbb{E}[U(L)] \geq \mathbb{E}[U(L')]$ *represents the preferences* (or ranking) $\succsim$.

Before we comment on Theorem 5.2.1 we give an example of how one can use it. Theorem 5.2.1 sometimes allows us to predict an individual's choice between two lotteries C and D if we know how that individual ranks two different lotteries A and B. For example, suppose that we observe that Susan is faced with the choice between lotteries A and B below and she says that she prefers A to B:

$$A = \begin{pmatrix} o_1 & o_2 & o_3 \\ 0 & 0.25 & 0.75 \end{pmatrix} \qquad B = \begin{pmatrix} o_1 & o_2 & o_3 \\ 0.2 & 0 & 0.8 \end{pmatrix}$$

With this information we can predict which of the following two lotteries C and D she will choose, if she has von Neumann-Morgenstern preferences:

$$C = \begin{pmatrix} o_1 & o_2 & o_3 \\ 0.8 & 0 & 0.2 \end{pmatrix} \qquad D = \begin{pmatrix} o_1 & o_2 & o_3 \\ 0 & 1 & 0 \end{pmatrix} = o_2.$$

Let U be a von Neumann-Morgenstern utility function whose existence is guaranteed by Theorem 5.2.1. Let $U(o_1) = a$, $U(o_2) = b$ and $U(o_3) = c$ (where a, b and c are numbers). Then, since Susan prefers A to B, the expected utility of A must be greater than the expected utility of B: $0.25b + 0.75c > 0.2a + 0.8c$. This inequality is equivalent to $0.25b > 0.2a + 0.05c$ or, dividing both sides by 0.25, $b > 0.8a + 0.2c$. It follows from this and Theorem 5.2.1 that Susan prefers D to C, because the expected utility of D is b and the expected utility of C is $0.8a + 0.2c$. Note that, in this example, we merely used the fact that a von Neumann-Morgenstern utility function *exists*, even though we do not know what the values of this function are.

Theorem 5.2.1 is an example of a "representation theorem" and is a generalization of a similar result for the case of the ranking of a finite set of basic outcomes O. It is not difficult to prove that if $\succsim$ is a complete and transitive ranking of O then there exists a function $U : O \to \mathbb{R}$, called a utility function (see Chapter 2), such that, for any two basic outcomes $o, o' \in O$, $U(o) \geq U(o')$ if and only if $o \succsim o'$. Now, it is quite possible that an individual has a complete and transitive ranking of O, is fully aware of her ranking and yet she is not able to answer the question "what is your utility function?", perhaps because she has never heard about utility functions. A utility function is a *tool* that we can use to represent her ranking, nothing more than that. The same applies to von Neumann-Morgenstern rankings: Theorem 5.2.1 tells us that if an individual has a von Neumann-Morgenstern ranking of the set of lotteries $\mathscr{L}(O)$ then there exists a von Neumann-Morgenstern utility function that we can use to represent her preferences, but it would not make sense for us to ask the individual "what is your von Neumann-Morgenstern utility function?" (indeed this was a question that could not even be conceived before von Neumann and Morgenstern stated and proved Theorem 5.2.1 in 1944!).

Theorem 5.2.1 tells us that a von Neumann-Morgenstern utility function exists; the next theorem can be used to actually construct such a function, by asking the individual to answer a few questions, formulated in a way that is fully comprehensible to her (without using the word 'utility'). The theorem says that, although there are many utility functions that represent a given von Neumann-Morgenstern ranking, once you know one function you "know them all", in the sense that there is a simple operation that transforms one function into the other.

Theorem 5.2.2 [von Neumann-Morgenstern, 1944].
Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathscr{L}(O)$, where $O = \{o_1, o_2, ..., o_m\}$. Then the following are true.

(A) If $U : O \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b, with $a > 0$, the function $V : O \to \mathbb{R}$ defined by $V(o_i) = aU(o_i) + b$ (for every $i = 1, \ldots, m$) is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U : O \to \mathbb{R}$ and $V : O \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b, with $a > 0$, such that $V(o_i) = aU(o_i) + b$ (for every $i = 1, \ldots, m$).

Proof. The proof of Part A of Theorem 5.2.2 is very simple. Let a and b be two numbers, with $a > 0$. The hypothesis is that $U : O \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, that is, that, for any two lotteries

$$L = \begin{pmatrix} o_1 & \cdots & o_m \\ p_1 & \cdots & p_m \end{pmatrix} \quad \text{and} \quad L' = \begin{pmatrix} o_1 & \cdots & o_m \\ q_1 & \cdots & q_m \end{pmatrix},$$

$$L \succsim L' \text{ if and only if } p_1U(o_1) + ... + p_mU(o_m) \geq q_1U(o_1) + ... + q_mU(o_m) \quad (5.1)$$

Multiplying both sides of the inequality in (5.1) by $a > 0$ and adding $(p_1 + \cdots + p_m)b$ to the left-hand side and $(q_1 + \cdots + q_m)b$ to the right-hand side we obtain

$$p_1[aU(o_1) + b] + ... + p_m[aU(o_m) + b] \geq q_1[aU(o_1) + b] + ... + q_m[aU(o_m) + b] \quad (5.2)$$

Defining $V(o_i) = aU(o_i) + b$, it follows from (5.1) and (5.2) that

$$L \succsim L' \text{ if and only if } p_1V(o_1) + ... + p_mV(o_m) \geq q_1V(o_1) + ... + q_mV(o_m),$$

that is, the function V is a von Neumann-Morgenstern utility function that represents the ranking $\succsim$. The proof of Part B will be given later, after introducing more notation and some observations. ■

Suppose that the *DM* has a von Neumann-Morgenstern ranking of the set of lotteries $\mathscr{L}(O)$. Since among the lotteries there are the degenerate ones that assign probability 1 to a single basic outcome, it follows that the *DM* has a complete and transitive ranking of the basic outcomes. We shall write o_{best} for a best basic outcome, that is, a basic outcome which is at least as good as any other basic outcome ($o_{best} \succsim o$, for every $o \in O$) and o_{worst} for a worst basic outcome, that is, a basic outcome such that every other outcome is at least as good as it ($o \succsim o_{worst}$, for every $o \in O$). Note that there may be several best outcomes (then the *DM* would be indifferent among them) and several worst outcomes; then o_{best} will denote an arbitrary best outcome and o_{worst} an arbitrary worst outcome. We shall assume throughout that the *DM* is not indifferent among all the outcomes, that is, we shall assume that $o_{best} \succ o_{worst}$.

We now show that, in virtue of Theorem 5.2.2, among the von Neumann-Morgenstern utility functions that represent a given von Neumann-Morgenstern ranking $\succsim$ of $\mathscr{L}(O)$, there is one that assigns the value 1 to the best basic outcome(s) and the value 0 to the worst basic outcome(s). To see this, consider an arbitrary von Neumann-Morgenstern

utility function $F : O \to \mathbb{R}$ that represents $\succsim$ and define $G : O \to \mathbb{R}$ as follows: for every $o \in O$, $G(o) = F(o) - F(o_{worst})$.
Then, by Theorem 5.2.2 (with $a = 1$ and $b = -F(o_{worst})$), G is also a utility function that represents $\succsim$ and, by construction, $G(o_{worst}) = F(o_{worst}) - F(o_{worst}) = 0$; note also that, since $o_{best} \succ o_{worst}$, it follows that $G(o_{best}) > 0$.
Finally, define $U : O \to \mathbb{R}$ as follows: for every $o \in O$, $U(o) = \frac{G(o)}{G(o_{best})}$.
Then, by Theorem 5.2.2 (with $a = \frac{1}{G(o_{best})}$ and $b = 0$), U is a utility function that represents $\succsim$ and, by construction, $U(o_{worst}) = 0$ and $U(o_{best}) = 1$.
For example, if there are six basic outcomes and the ranking of the basic outcomes is $o_3 \sim o_6 \succ o_1 \succ o_4 \succ o_2 \sim o_5$, then one can take as o_{best} either o_3 or o_6 and as o_{worst} either o_2 or o_5; furthermore, if F is given by $\begin{array}{cccccc} o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\ 2 & -2 & 8 & 0 & -2 & 8 \end{array}$ then G is the function $\begin{array}{cccccc} o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\ 4 & 0 & 10 & 2 & 0 & 10 \end{array}$ and U is the function $\begin{array}{cccccc} o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\ 0.4 & 0 & 1 & 0.2 & 0 & 1 \end{array}$.

Definition 5.2.2 Let $U : O \to \mathbb{R}$ be a utility function that represents a given von Neumann-Morgenstern ranking $\succsim$ of the set of lotteries $\mathscr{L}(O)$. We say that U is *normalized* if $U(o_{worst}) = 0$ and $U(o_{best}) = 1$.

The transformations described above show how to normalize any given utility function. Armed with the notion of a normalized utility function we can now complete the proof of Theorem 5.2.2.

Proof of Part B of Theorem 5.2.2. Let $F : O \to \mathbb{R}$ and $G : O \to \mathbb{R}$ be two von Neumann-Morgenstern utility functions that represent a given von Neumann-Morgenstern ranking of $\mathscr{L}(O)$.
Let $U : O \to \mathbb{R}$ be the normalization of F and $V : O \to \mathbb{R}$ be the normalization of G. First we show that it must be that $U = V$, that is, $U(o) = V(o)$ for every $o \in O$.
Suppose, by contradiction, that there is an $\hat{o} \in O$ such that $U(\hat{o}) \neq V(\hat{o})$. Without loss of generality we can assume that $U(\hat{o}) > V(\hat{o})$.
Construct the following lottery: $L = \begin{pmatrix} o_{best} & o_{worst} \\ \hat{p} & 1-\hat{p} \end{pmatrix}$ with $\hat{p} = U(\hat{o})$ (recall that U is normalized and thus takes on values in the interval from 0 to 1).
Then $\mathbb{E}[U(L)] = \mathbb{E}[V(L)] = U(\hat{o})$. Hence, according to U it must be that $\hat{o} \sim L$ (this follows from Theorem 5.2.1), while according to V it must be (again, by Theorem 5.2.1) that $L \succ \hat{o}$ (since $\mathbb{E}[V(L)] = U(\hat{o}) > V(\hat{o})$). Then U and V cannot be two representations of the same ranking. Now let $a_1 = \frac{1}{F(o_{best}) - F(o_{worst})}$ and $b_1 = -\frac{F(o_{worst})}{F(o_{best}) - F(o_{worst})}$.
Note that $a_1 > 0$. Then it is easy to verify that, for every $o \in O$, $U(o) = a_1 F(o) + b_1$.
Similarly let $a_2 = \frac{1}{G(o_{best}) - G(o_{worst})}$ and $b_2 = -\frac{G(o_{worst})}{G(o_{best}) - G(o_{worst})}$; again, $a_2 > 0$ and, for every $o \in O$, $V(o) = a_2 G(o) + b_2$. We can invert the latter transformation and obtain that, for every $o \in O$, $G(o) = \frac{V(o)}{a_2} - \frac{b_2}{a_2}$.
Thus, we can transform F into U, which – as proved above – is the same as V, and then transform V into G thus obtaining the following transformation of F into G: $G(o) = aF(o) + b$ where $a = \frac{a_1}{a_2} > 0$ and $b = \frac{b_1 - b_2}{a_2}$. ■

R Theorem 5.2.2 is often stated as follows: a utility function that represents a von Neumann-Morgenstern ranking $\succsim$ of $\mathscr{L}(O)$ is *unique up to a positive affine transformation*.[2] Because of Theorem 5.2.2, a von Neumann-Morgenstern utility function is usually referred to as a *cardinal* utility function.

Theorem 5.2.1 guarantees the existence of a utility function that represents a given von Neumann-Morgenstern ranking $\succsim$ of $\mathscr{L}(O)$ and Theorem 5.2.2 characterizes the set of such functions. Can one actually construct a utility function that represents a given ranking? The answer is affirmative: if there are m basic outcomes one can construct an individual's von Neumann-Morgenstern utility function by asking her at most $(m-1)$ questions. The first question is "what is your ranking of the basic outcomes?". Then we can construct the normalized utility function by first assigning the value 1 to the best outcome(s) and the value 0 to the worst outcome(s). This leaves us with at most $(m-2)$ values to determine. For this we appeal to one of the axioms discussed in the next section, namely the Continuity Axiom, which says that, for every basic outcome o_i there is a probability $p_i \in [0,1]$ such that the *DM* is indifferent between o_i for certain and the lottery that gives a best outcome with probability p_i and a worst outcome with probability $(1-p_i)$: $o_i \sim \begin{pmatrix} o_{best} & o_{worst} \\ p_i & 1-p_i \end{pmatrix}$. Thus, for each basic outcome o_i for which a utility has not been determined yet, we should ask the individual to tell us the value of p_i such that $o_i \sim \begin{pmatrix} o_{best} & o_{worst} \\ p_i & 1-p_i \end{pmatrix}$; then we can set $U_i(o_i) = p_i$, because the expected utility of the lottery $\begin{pmatrix} o_{best} & o_{worst} \\ p_i & 1-p_i \end{pmatrix}$ is $p_iU_i(o_{best}) + (1-p_i)U_i(o_{worst}) = p_i(1) + (1-p_i)0 = p_i$.

■ **Example 5.1** Suppose that there are five basic outcomes, that is, $O = \{o_1, o_2, o_3, o_4, o_5\}$ and the *DM*, who has von Neumann-Morgenstern preferences, tells us that her ranking of the basic outcomes is as follows: $o_2 \succ o_1 \sim o_5 \succ o_3 \sim o_4$.

- Then we can begin by assigning utility 1 to the best outcome o_2 and utility 0 to the worst outcomes o_3 and o_4: $\begin{pmatrix} \text{outcome:} & o_1 & o_2 & o_3 & o_4 & o_5 \\ \text{utility:} & ? & 1 & 0 & 0 & ? \end{pmatrix}$.
- There is only one value left to be determined, namely the utility of o_1 (which is also the utility of o_5, since $o_1 \sim o_5$).
- To find this value, we ask the *DM* to tell us what value of p makes her indifferent between the lottery $L = \begin{pmatrix} o_2 & o_3 \\ p & 1-p \end{pmatrix}$ and outcome o_1 with certainty.
- Suppose that her answer is: 0.4. Then her normalized von Neumann-Morgenstern utility function is $\begin{pmatrix} \text{outcome:} & o_1 & o_2 & o_3 & o_4 & o_5 \\ \text{utility:} & 0.4 & 1 & 0 & 0 & 0.4 \end{pmatrix}$. Knowing this, we can predict her choice among any set of lotteries over these five basic outcomes.

■

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 5.4.2 at the end of this chapter.

[2] An affine transformation is a function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ with $a, b \in \mathbb{R}$. The affine transformation is positive if $a > 0$.

5.3 Expected utility: the axioms

We can now turn to the list of rationality axioms proposed by von Neumann and Morgenstern. This section makes heavy use of mathematical notation and, as mentioned in the previous section, if the reader is not interested in understanding in what sense the theory of expected utility captures the notion of rationality, he/she can skip it without affecting his/her ability to understand the rest of this book.

Let $O = \{o_1, o_2, ..., o_m\}$ be the set of basic outcomes and $\mathscr{L}(O)$ the set of simple lotteries, that is, the set of probability distributions over O. Let $\succsim$ be a binary relation on $\mathscr{L}(O)$. We say that $\succsim$ is a von Neumann-Morgenstern ranking of $\mathscr{L}(O)$ if it satisfies the following four axioms or properties.

Axiom 1 [Completeness and transitivity]. $\succsim$ is complete (for every two lotteries L and L' either $L \succsim L'$ or $L' \succsim L$ or both) and transitive (for any three lotteries L_1, L_2 and L_3, if $L_1 \succsim L_2$ and $L_2 \succsim L_3$ then $L_1 \succsim L_3$).

As noted in the previous section, Axiom 1 implies that there is a complete and transitive ranking of the basic outcomes. Recall that o_{best} denotes a best basic outcome and o_{worst} denotes a worst basic outcome and that we are assuming that $o_{best} \succ o_{worst}$, that is, that the *DM* is not indifferent among all the basic outcomes.

Axiom 2 [Monotonicity]. $\begin{pmatrix} o_{best} & o_{worst} \\ p & 1-p \end{pmatrix} \succsim \begin{pmatrix} o_{best} & o_{worst} \\ q & 1-q \end{pmatrix}$ if and only if $p \geq q$.

Axiom 3 [Continuity]. For every basic outcome o_i there is a $p_i \in [0,1]$ such that $o_i \sim \begin{pmatrix} o_{best} & o_{worst} \\ p_i & 1-p_i \end{pmatrix}$.

Before we introduce the last axiom we need to define a compound lottery.

Definition 5.3.1 A *compound lottery* is a lottery of the form $\begin{pmatrix} x_1 & x_2 & \dots & x_r \\ p_1 & p_2 & \dots & p_r \end{pmatrix}$ where each x_i is either an element of O or an element of $\mathscr{L}(O)$.

For example, let $m = 4$.

Then $L = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{2}{5} & 0 & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$ is a simple lottery (an element of $\mathscr{L}(O)$),

while $C = \begin{pmatrix} \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix} & o_1 & \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{pmatrix} \\ & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ is a compound lottery.[3]

[3] With $r = 3$, $x_1 = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$, $x_2 = o_1$, $x_3 = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{pmatrix}$, $p_1 = \frac{1}{2}, p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{4}$.

The compound lottery $C = \begin{pmatrix} \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix} & o_1 & \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{pmatrix} \\ \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

can be viewed graphically as a tree, as shown in Figure 5.1.

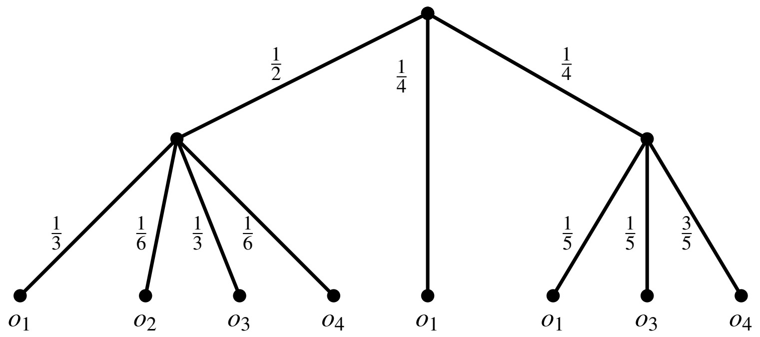

Figure 5.1: A compound lottery

Definition 5.3.2 Given a compound lottery $C = \begin{pmatrix} x_1 & x_2 & \ldots & x_r \\ p_1 & p_2 & \ldots & p_r \end{pmatrix}$ the *corresponding simple lottery* $L(C) = \begin{pmatrix} o_1 & o_2 & \ldots & o_m \\ q_1 & q_2 & \ldots & q_m \end{pmatrix}$ is defined as follows. First of all, for $i = 1, \ldots, m$ and $j = 1, \ldots, r$, define

$$o_i(x_j) = \begin{cases} 1 & \text{if } x_j = o_i \\ 0 & \text{if } x_j = o_k \text{ with } k \neq i \\ s_i & \text{if } x_j = \begin{pmatrix} o_1 & \ldots & o_{i-1} & o_i & o_{i+1} & \ldots & o_m \\ s_1 & \ldots & s_{i-1} & s_i & s_{i+1} & \ldots & s_m \end{pmatrix} \end{cases}$$

Then $q_i = \sum_{j=1}^{r} p_j\, o_i(x_j)$.

Continuing the above example where

$$C = \begin{pmatrix} \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix} & o_1 & \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{pmatrix} \\ \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(see Figure 5.1) we have that $r = 3$, $x_1 = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$, $x_2 = o_1$ and $x_3 = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{pmatrix}$, so that

$$o_1(x_1) = \tfrac{1}{3}, \quad o_1(x_2) = 1, \quad \text{and} \quad o_1(x_3) = \tfrac{1}{5}$$

and thus $q_1 = \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{4}(1) + \frac{1}{4}\left(\frac{1}{5}\right) = \frac{28}{60}$. Similarly, $q_2 = \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{4}(0) + \frac{1}{4}(0) = \frac{1}{12} = \frac{5}{60}$,

$q_3 = \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{4}(0) + \frac{1}{4}\left(\frac{1}{5}\right) = \frac{13}{60}$ and $q_4 = \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{4}(0) + \frac{1}{4}\left(\frac{3}{5}\right) = \frac{14}{60}$. These numbers correspond to multiplying the probabilities along the edges of the tree of Figure 5.1 leading to an outcome, as shown in Figure 5.2 and then adding up the probabilities of each outcome, as shown in Figure 5.3. Thus, the simple lottery $L(C)$ that corresponds to C is $L(C) = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{28}{60} & \frac{5}{60} & \frac{13}{60} & \frac{14}{60} \end{pmatrix}$, namely the lottery shown in Figure 5.3.

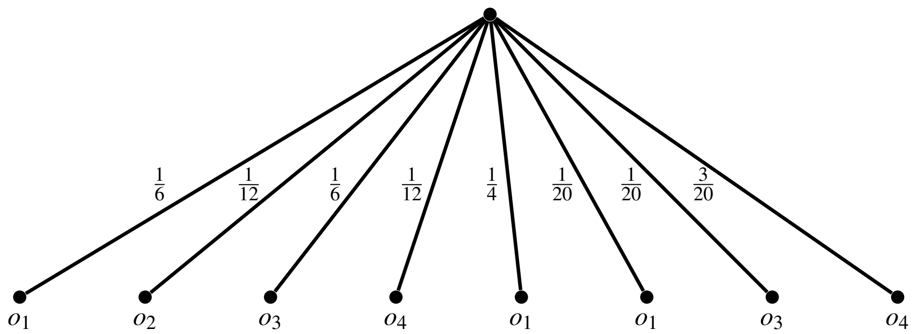

Figure 5.2: Simplification of Figure 5.1 obtained by merging paths into simple edges and associating with the simple edges the products of the probabilities along the path.

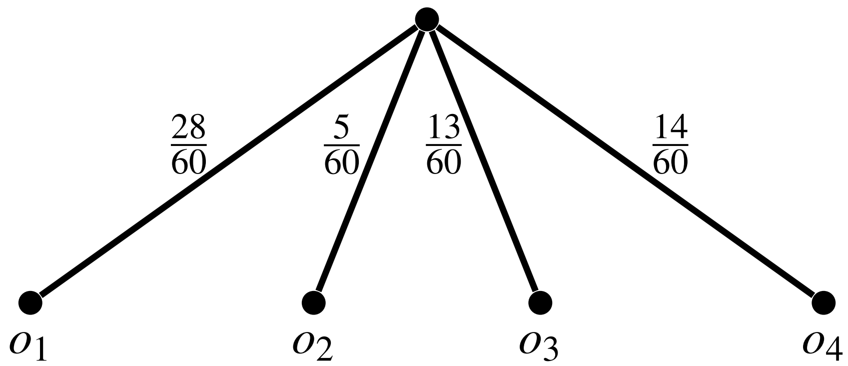

Figure 5.3: Simplification of Figure 5.2 obtained by adding, for each outcome, the probabilities of that outcome.

Axiom 4 [Independence or substitutability]. Consider an arbitrary basic outcome o_i and an arbitrary simple lottery $L = \begin{pmatrix} o_1 & \cdots & o_{i-1} & o_i & o_{i+1} & \cdots & o_m \\ p_1 & \cdots & p_{i-1} & p_i & p_{i+1} & \cdots & p_m \end{pmatrix}$. If $\hat{L}$ is a simple lottery such that $o_i \sim \hat{L}$, then $L \sim M$ where M is the simple lottery corresponding to the compound lottery $C = \begin{pmatrix} o_1 & \cdots & o_{i-1} & \hat{L} & o_{i+1} & \cdots & o_m \\ p_1 & \cdots & p_{i-1} & p_i & p_{i+1} & \cdots & p_m \end{pmatrix}$ obtained by replacing o_i with $\hat{L}$ in L.

We can now prove the first theorem of the previous section.

Proof of Theorem 5.2.1. To simplify the notation, throughout this proof we will assume that we have renumbered the basic outcomes in such a way that $o_{best} = o_1$ and $o_{worst} = o_m$. First of all, for every basic outcome o_i, let $u_i \in [0,1]$ be such that $o_i \sim \begin{pmatrix} o_1 & o_m \\ u_i & 1-u_i \end{pmatrix}$. The existence of such a value u_i is guaranteed by the Continuity Axiom (Axiom 3); clearly $u_1 = 1$ and $u_m = 0$. Now consider an arbitrary lottery

$$L_1 = \begin{pmatrix} o_1 & \dots & o_m \\ p_1 & \dots & p_m \end{pmatrix}.$$

First we show that

$$L_1 \sim \begin{pmatrix} o_1 & o_m \\ \sum_{i=1}^{m} p_i u_i & 1 - \sum_{i=1}^{m} p_i u_i \end{pmatrix} \tag{5.3}$$

This is done through a repeated application of the Independence Axiom (Axiom 4), as follows. Consider the compound lottery

$$\mathscr{C}_2 = \begin{pmatrix} o_1 & \begin{pmatrix} o_1 & o_m \\ u_2 & 1-u_2 \end{pmatrix} & o_3 & \dots & o_m \\ p_1 & p_2 & p_3 & \dots & p_m \end{pmatrix}$$

obtained by replacing o_2 in lottery L_1 with the lottery $\begin{pmatrix} o_1 & o_m \\ u_2 & 1-u_2 \end{pmatrix}$ that the *DM* considers to be just as good as o_2. The simple lottery corresponding to $\mathscr{C}_2$ is

$$L_2 = \begin{pmatrix} o_1 & o_3 & \dots & o_{m-1} & o_m \\ p_1 + p_2 u_2 & p_3 & \dots & p_{m-1} & p_m + p_2(1-u_2) \end{pmatrix}.$$

Note that o_2 is assigned probability 0 in L_2 and thus we have omitted it. By Axiom 4, $L_1 \sim L_2$. Now apply the same argument to L_2: let

$$\mathscr{C}_3 = \begin{pmatrix} o_1 & \begin{pmatrix} o_1 & o_m \\ u_3 & 1-u_3 \end{pmatrix} & \dots & o_m \\ p_1 + p_2 u_2 & p_3 & \dots & p_m + p_2(1-u_2) \end{pmatrix}$$

whose corresponding simple lottery is

$$L_3 = \begin{pmatrix} o_1 & \dots & o_m \\ p_1 + p_2 u_2 + p_3 u_3 & \dots & p_m + p_2(1-u_2) + p_3(1-u_3) \end{pmatrix}.$$

Note, again, that o_3 is assigned probability zero in L_3. By Axiom 4, $L_2 \sim L_3$; thus, by transitivity (since $L_1 \sim L_2$ and $L_2 \sim L_3$) we have that $L_1 \sim L_3$. Repeating this argument we get that $L_1 \sim L_{m-1}$, where

$$L_{m-1} = \begin{pmatrix} o_1 & o_m \\ p_1 + p_2 u_2 + \dots + p_{m-1} u_{m-1} & p_m + p_2(1-u_2) + \dots + p_{m-1}(1-u_{m-1}) \end{pmatrix}.$$

Since $u_1 = 1$ (so that $p_1 u_1 = p_1$) and $u_m = 0$ (so that $p_m u_m = 0$),

$$p_1 + p_2 u_2 + \ldots + p_{m-1} u_{m-1} = \sum_{i=1}^{m} p_i u_i$$

and

$$p_2(1-u_2) + \ldots + p_{m-1}(1-u_{m-1}) + p_m = \sum_{i=2}^{m} p_i - \sum_{i=2}^{m-1} p_i u_i = p_1 + \sum_{i=2}^{m} p_i - \sum_{i=2}^{m-1} p_i u_i - p_1$$

$$= _{(\text{since } u_1=1 \text{ and } u_m=0)} \sum_{i=1}^{m} p_i - \sum_{i=2}^{m-1} p_i u_i - p_1 u_1 - p_m u_m = _{\left(\text{since } \sum_{i=1}^{m} p_i = 1\right)} 1 - \sum_{i=1}^{m} p_i u_i$$

Thus, $L_{m-1} = \begin{pmatrix} o_1 & o_m \\ \sum_{i=1}^{m} p_i u_i & 1 - \sum_{i=1}^{m} p_i u_i \end{pmatrix}$, which proves (5.3). Now define the following utility function $U : \{o_1, \ldots, o_m\} \to [0,1]$: $U(o_i) = u_i$, where, as before, for every basic outcome o_i, $u_i \in [0,1]$ is such that $o_i \sim \begin{pmatrix} o_1 & o_m \\ u_i & 1-u_i \end{pmatrix}$. Consider two arbitrary lotteries $L = \begin{pmatrix} o_1 & \ldots & o_m \\ p_1 & \ldots & p_m \end{pmatrix}$ and $L' = \begin{pmatrix} o_1 & \ldots & o_m \\ q_1 & \ldots & q_m \end{pmatrix}$. We want to show that $L \succsim L'$ if and only if $\mathbb{E}[U(L)] \geq \mathbb{E}[U(L')]$, that is, if and only if $\sum_{i=1}^{m} p_i u_i \geq \sum_{i=1}^{m} q_i u_i$. By (5.3), $L \sim M$, where $M = \begin{pmatrix} o_1 & o_m \\ \sum_{i=1}^{m} p_i u_i & 1 - \sum_{i=1}^{m} p_i u_i \end{pmatrix}$ and also $L' \sim M'$, where $M' = \begin{pmatrix} o_1 & o_m \\ \sum_{i=1}^{m} q_i u_i & 1 - \sum_{i=1}^{m} q_i u_i \end{pmatrix}$. Thus, by transitivity of $\succsim$, $L \succsim L'$ if and only if $M \succsim M'$; by the Monotonicity Axiom (Axiom 2), $M \succsim M'$ if and only if $\sum_{i=1}^{m} p_i u_i \geq \sum_{i=1}^{m} q_i u_i$. ■

The following example, known as the *Allais paradox*, suggests that one should view expected utility theory as a "prescriptive" or "normative" theory (that is, as a theory about how rational people should choose) rather than as a descriptive theory (that is, as a theory about the actual behavior of individuals). In 1953 the French economist Maurice Allais published a paper regarding a survey he had conducted in 1952 concerning a hypothetical decision problem. Subjects "with good training in and knowledge of the theory of probability, so that they could be considered to behave rationally" were asked to rank the following pairs of lotteries:

$$A = \begin{pmatrix} \$5 \text{ Million} & \$0 \\ \frac{89}{100} & \frac{11}{100} \end{pmatrix} \quad \textit{versus} \quad B = \begin{pmatrix} \$1 \text{ Million} & \$0 \\ \frac{90}{100} & \frac{10}{100} \end{pmatrix}$$

and

$$C = \begin{pmatrix} \$5 \text{ Million} & \$1 \text{ Million} & \$0 \\ \frac{89}{100} & \frac{10}{100} & \frac{1}{100} \end{pmatrix} \quad \textit{versus} \quad D = \begin{pmatrix} \$1 \text{ Million} \\ 1 \end{pmatrix}.$$

Most subjects reported the following ranking: $A \succ B$ and $D \succ C$. Such ranking violates the axioms of expected utility. To see this, let $O = \{o_1, o_2, o_3\}$ with $o_1 = \$5$ Million, $o_2 = \$1$ Million and $o_3 = \$0$. Let us assume that the individual in question prefers more money to less, so that $o_1 \succ o_2 \succ o_3$ and has a von Neumann-Morgenstern ranking of the lotteries over $\mathscr{L}(O)$. Let $u_2 \in (0,1)$ be such that $D \sim \begin{pmatrix} \$5 \text{ Million} & \$0 \\ u_2 & 1-u_2 \end{pmatrix}$ (the existence of such u_2 is guaranteed by the Continuity Axiom). Then, since $D \succ C$, by transitivity

$$\begin{pmatrix} \$5 \text{ Million} & \$0 \\ u_2 & 1-u_2 \end{pmatrix} \succ C. \tag{5.4}$$

Let C' be the simple lottery corresponding to the compound lottery

$$\begin{pmatrix} \$5 \text{ Million} & \begin{pmatrix} \$5 \text{ Million} & \$0 \\ u_2 & 1-u_2 \end{pmatrix} & \$0 \\ \frac{89}{100} & \frac{10}{100} & \frac{1}{100} \end{pmatrix}.$$

Then $C' = \begin{pmatrix} \$5 \text{ Million} & \$0 \\ \frac{89}{100} + \frac{10}{100}u_2 & 1 - \left(\frac{89}{100} + \frac{10}{100}u_2\right) \end{pmatrix}$.

By the Independence Axiom, $C \sim C'$ and thus, by (5.4) and transitivity,

$$\begin{pmatrix} \$5 \text{ Million} & \$0 \\ u_2 & 1-u_2 \end{pmatrix} \succ \begin{pmatrix} \$5 \text{ Million} & \$0 \\ \frac{89}{100} + \frac{10}{100}u_2 & 1 - \left(\frac{89}{100} + \frac{10}{100}u_2\right) \end{pmatrix}.$$

Hence, by the Monotonicity Axiom, $u_2 > \frac{89}{100} + \frac{10}{100}u_2$, that is,

$$u_2 > \frac{89}{90}. \tag{5.5}$$

Let B' be the simple lottery corresponding to the following compound lottery, constructed from B by replacing the basic outcome '\$1 Million' with $\begin{pmatrix} \$5 \text{ Million} & \$0 \\ u_2 & 1-u_2 \end{pmatrix}$:

$$\begin{pmatrix} \begin{pmatrix} \$5\, Million & \$0 \\ u_2 & 1-u_2 \end{pmatrix} & \$0 \\ \frac{90}{100} & \frac{10}{100} \end{pmatrix}.$$

Then

$$B' = \begin{pmatrix} \$5 \text{ Million} & \$0 \\ \frac{90}{100}u_2 & 1 - \frac{90}{100}u_2 \end{pmatrix}.$$

By the Independence Axiom, $B \sim B'$; thus, since $A \succ B$, by transitivity, $A \succ B'$ and therefore, by the Monotonicity Axiom, $\frac{89}{100} > \frac{90}{100}u_2$, that is, $u_2 < \frac{89}{90}$, contradicting (5.5).

Thus, if one finds the expected utility axioms compelling as axioms of rationality, then one cannot consistently express a preference for A over B and also a preference for D over C.

Another well-known example is the *Ellsberg paradox*. Suppose that you are told that an urn contains 30 red balls and 60 more balls that are either blue or yellow. You don't know how many blue or how many yellow balls there are, but the number of blue balls plus the number of yellow ball equals 60 (they could be all blue or all yellow or any combination of the two). The balls are well mixed so that each individual ball is as likely to be drawn as any other. You are given a choice between the bets A and B, where

A = you get \$100 if you pick a red ball and nothing otherwise,
B = you get \$100 if you pick a blue ball and nothing otherwise.

Many subjects in experiments state a strict preference for A over B: $A \succ B$. Consider now the following bets:

C = you get \$100 if you pick a red or yellow ball and nothing otherwise,
D = you get \$100 if you pick a blue or yellow ball and nothing otherwise.

Do the axioms of expected utility constrain your ranking of C and D? Many subjects in experiments state the following ranking: $A \succ B$ and $D \succsim C$. All such people violate the axioms of expected utility. The fraction of red balls in the urn is $\frac{30}{90} = \frac{1}{3}$. Let p_2 be the fraction of blue balls and p_3 the fraction of yellow balls (either of these can be zero: all we know is that $p_2 + p_3 = \frac{60}{90} = \frac{2}{3}$). Then A, B, C and D can be viewed as the following lotteries:

$$A = \begin{pmatrix} \$100 & \$0 \\ \frac{1}{3} & p_2 + p_3 \end{pmatrix}, \quad B = \begin{pmatrix} \$100 & \$0 \\ p_2 & \frac{1}{3} + p_3 \end{pmatrix}$$

$$C = \begin{pmatrix} \$100 & \$0 \\ \frac{1}{3} + p_3 & p_2 \end{pmatrix}, \quad D = \begin{pmatrix} \$100 & \$0 \\ p_2 + p_3 = \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Let U be the normalized von Neumann-Morgenstern utility function that represents the individual's ranking; then $U(\$100) = 1$ and $U(0) = 0$. Thus,

$$\mathbb{E}[U(A)] = \tfrac{1}{3}, \quad \mathbb{E}[U(B)] = p_2, \quad \mathbb{E}[U(C)] = \tfrac{1}{3} + p_3, \quad \text{and} \quad \mathbb{E}[U(D)] = p_2 + p_3 = \tfrac{2}{3}.$$

Hence, $A \succ B$ if and only if $\frac{1}{3} > p_2$, which implies that $p_3 > \frac{1}{3}$, so that $\mathbb{E}[U(C)] = \frac{1}{3} + p_3 > \mathbb{E}[U(D)] = \frac{2}{3}$ and thus $C \succ D$ (similarly, $B \succ A$ if and only if $\frac{1}{3} < p_2$, which implies that $\mathbb{E}[U(C)] < \mathbb{E}[U(D)]$ and thus $D \succ C$).

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 5.4.3 at the end of this chapter.

5.4 Exercises

The solutions to the following exercises are given in Section 5.5 at the end of this chapter.

5.4.1 Exercises for Section 5.1: Money lotteries and attitudes to risk

Exercise 5.1 What is the expected value of the following lottery?

$$\begin{pmatrix} 24 & 12 & 48 & 6 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}$$

■

Exercise 5.2 Consider the following lottery:

$$\begin{pmatrix} o_1 & o_2 & o_3 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

where
- o_1 = you get an invitation to have dinner at the White House,
- o_2 = you get (for free) a puppy of your choice
- o_3 = you get \$600.

What is the expected value of this lottery? ■

Exercise 5.3 Consider the following money lottery

$$L = \begin{pmatrix} \$10 & \$15 & \$18 & \$20 & \$25 & \$30 & \$36 \\ \frac{3}{12} & \frac{1}{12} & 0 & \frac{3}{12} & \frac{2}{12} & 0 & \frac{3}{12} \end{pmatrix}$$

(a) What is the expected value of the lottery?
(b) Ann prefers more money to less and has transitive preferences. She says that, between getting \$20 for certain and playing the above lottery, she would prefer \$20 for certain. What is her attitude to risk?
(c) Bob prefers more money to less and has transitive preferences. He says that, given the same choice as Ann, he would prefer playing the lottery. What is his attitude to risk?

■

Exercise 5.4 Sam has a debilitating illness and has been offered two mutually exclusive courses of action: (1) take some well-known drugs which have been tested for a long time and (2) take a new experimental drug. If he chooses (1) then for certain his pain will be reduced to a bearable level. If he chooses (2) then he has a 50% chance of being completely cured and a 50% chance of no benefits from the drug and possibly some harmful side effects. He chose (1). What is his attitude to risk? ■

5.4.2 Exercises for Section 5.2: Expected utility theory

Exercise 5.5 Ben is offered a choice between the following two money lotteries:
$A = \begin{pmatrix} \$4,000 & \$0 \\ 0.8 & 0.2 \end{pmatrix}$ and $B = \begin{pmatrix} \$3,000 \\ 1 \end{pmatrix}$. He says he strictly prefers B to A. Which of the following two lotteries, C and D, will Ben choose if he satisfies the axioms of expected utility and prefers more money to less?
$C = \begin{pmatrix} \$4,000 & \$0 \\ 0.2 & 0.8 \end{pmatrix}$, $D = \begin{pmatrix} \$3,000 & \$0 \\ 0.25 & 0.75 \end{pmatrix}$. ■

Exercise 5.6 There are three basic outcomes, o_1, o_2 and o_3. Ann satisfies the axioms of expected utility and her preferences over lotteries involving these three outcomes can be represented by the following von Neumann-Morgenstern utility function:
$V(o_2) = a > V(o_1) = b > V(o_3) = c$. Normalize the utility function. ■

Exercise 5.7 Consider the following lotteries:

$$L_1 = \begin{pmatrix} \$3000 & \$500 \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}, \quad L_2 = \begin{pmatrix} \$3000 & \$500 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

$$L_3 = \begin{pmatrix} \$3000 & \$2000 & \$1000 & \$500 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad L_4 = \begin{pmatrix} \$2000 & \$1000 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Jennifer says that she is indifferent between lottery L_1 and getting \$2,000 for certain. She is also indifferent between lottery L_2 and getting \$1,000 for certain. Finally, she says that between L_3 and L_4 she would chose L_3.
Is she rational according to the theory of expected utility? [Assume that she prefers more money to less.] ■

Exercise 5.8 Consider the following basic outcomes:

- o_1 = a Summer internship at the White House,
- o_2 = a free one-week vacation in Europe,
- o_3 = \$800,
- o_4 = a free ticket to a concert.

Rachel says that her ranking of these outcomes is $o_1 \succ o_2 \succ o_3 \succ o_4$. She also says that **(1)** she is indifferent between $\begin{pmatrix} o_2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} o_1 & o_4 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$ and **(2)** she is indifferent between $\begin{pmatrix} o_3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} o_1 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. If she satisfies the axioms of expected utility theory, which of the two lotteries $L_1 = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{8} & \frac{2}{8} & \frac{3}{8} & \frac{2}{8} \end{pmatrix}$ and $L_2 = \begin{pmatrix} o_1 & o_2 & o_3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$ will she choose? ■

Exercise 5.9 Consider the following lotteries: $L_1 = \begin{pmatrix} \$30 & \$28 & \$24 & \$18 & \$8 \\ \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} \end{pmatrix}$ and $L_2 = \begin{pmatrix} \$30 & \$28 & \$8 \\ \frac{1}{10} & \frac{4}{10} & \frac{5}{10} \end{pmatrix}$.

(a) Which lottery would a risk neutral person choose?

(b) Paul's von Neumann-Morgenstern utility-of-money function is $U(m) = ln(m)$, where ln denotes the natural logarithm. Which lottery would Paul choose?

■

Exercise 5.10 There are five basic outcomes. Jane has a von Neumann-Morgenstern ranking of the set of lotteries over the set of basic outcomes that can be represented by either of the following utility functions U and V: $\begin{pmatrix} & o_1 & o_2 & o_3 & o_4 & o_5 \\ U: & 44 & 170 & -10 & 26 & 98 \\ V: & 32 & 95 & 5 & 23 & 59 \end{pmatrix}$.

(a) Show how to normalize each of U and V and verify that you get the same normalized utility function.

(b) Show how to transform U into V with a positive affine transformation of the form $x \mapsto ax + b$ with $a, b \in \mathbb{R}$ and $a > 0$.

■

Exercise 5.11 Consider the following lotteries: $L_3 = \begin{pmatrix} \$28 \\ 1 \end{pmatrix}$, $L_4 = \begin{pmatrix} \$10 & \$50 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(a) Ann has the following von Neumann-Morgenstern utility function: $U_{Ann}(\$m) = \sqrt{m}$. How does she rank the two lotteries?

(b) Bob has the following von Neumann-Morgenstern utility function: $U_{Bob}(\$m) = 2m - \frac{m^4}{100^3}$. How does he rank the two lotteries?

(c) Verify that both Ann and Bob are risk averse, by determining what they would choose between lottery L_4 and its expected value for certain.

■

5.4.3 Exercises for Section 5.3: Expected utility axioms

Exercise 5.12 Let $O = \{o_1, o_2, o_3, o_4\}$. Find the simple lottery corresponding to the following compound lottery

$$\begin{pmatrix} \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{2}{5} & \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \end{pmatrix} & o_2 & \begin{pmatrix} o_1 & o_3 & o_4 \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \end{pmatrix} & \begin{pmatrix} o_2 & o_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{2} \end{pmatrix}$$

■

Exercise 5.13 Let $O = \{o_1, o_2, o_3, o_4\}$. Suppose that the *DM* has a von Neumann-Morgenstern ranking of $\mathscr{L}(O)$ and states the following indifference:
$o_1 \sim \begin{pmatrix} o_2 & o_4 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ and $o_2 \sim \begin{pmatrix} o_3 & o_4 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$.

Find a lottery that the *DM* considers just as good as $L = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{2}{9} & \frac{1}{9} & \frac{1}{3} \end{pmatrix}$.

Do not add any information to what is given above (in particular, do not make any assumptions about which outcome is best and which is worst). ■

Exercise 5.14 — ⋆⋆⋆ Challenging Question ⋆⋆⋆.
Would you be willing to pay more in order to reduce the probability of dying within the next hour from one sixth to zero or from four sixths to three sixths? Unfortunately, this is not a hypothetical question: you accidentally entered the office of a mad scientist and have been overpowered and tied to a chair. The mad scientist has put six glasses in front of you, numbered 1 to 6, and tells you that one of them contains a deadly poison and the other five contain a harmless liquid. He says that he is going to roll a die and make you drink from the glass whose number matches the number that shows from the rolling of the die. You beg to be exempted and he asks you "what is the largest amount of money that you would be willing to pay to replace the glass containing the poison with one containing a harmless liquid?". Interpret this question as "what sum of money x makes you indifferent between (1) leaving the poison in whichever glass contains it and rolling the die, and (2) reducing your wealth by $\$x$ and rolling the die after the poison has been replaced by a harmless liquid". Your answer is: $\$X$. Then he asks you "suppose that instead of one glass with poison there had been four glasses with poison (and two with a harmless liquid); what is the largest amount of money that you would be willing to pay to replace one glass with poison with a glass containing a harmless liquid (and thus roll the die with 3 glasses with poison and 3 with a harmless liquid)?". Your answer is: $\$Y$. Show that if $X > Y$ then you do not satisfy the axioms of Expected Utility Theory. [Hint: think about what the basic outcomes are; assume that you do not care about how much money is left in your estate if you die and that, when alive, you prefer more money to less.] ■

5.5 Solutions to Exercises

Solution to Exercise 5.1 The expected value of the lottery $\begin{pmatrix} 24 & 12 & 48 & 6 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}$ is
$\frac{1}{6}(24) + \frac{2}{6}(12) + \frac{1}{6}(48) + \frac{2}{6}(6) = 18.$ □

Solution to Exercise 5.2 This was a trick question! There is no expected value because the basic outcomes are not numbers. □

Solution to Exercise 5.3

(a) The expected value of the lottery

$$L = \begin{pmatrix} \$10 & \$15 & \$18 & \$20 & \$25 & \$30 & \$36 \\ \frac{3}{12} & \frac{1}{12} & 0 & \frac{3}{12} & \frac{2}{12} & 0 & \frac{3}{12} \end{pmatrix}$$

is $\mathbb{E}[L] = \frac{3}{12}(10) + \frac{1}{12}(15) + (0)(18) + \frac{3}{12}(20) + \frac{2}{12}(25) + (0)(30) + \frac{3}{12}(36) = \frac{263}{12} =$ \$21.92

(b) Since Ann prefers more money to less, she prefers \$21.92 to \$20 (\$21.92 $\succ$ \$20). She said that she prefers \$20 to lottery L (\$20 $\succ L$). Thus, since her preferences are transitive, she prefers \$21.92 to lottery L (\$21.92 $\succ L$). Hence, she is risk averse.

(c) The answer is: we cannot tell. First of all, since Bob prefers more money to less, he prefers \$21.92 to \$20 (\$21.92 $\succ$ \$20). Bob could be risk neutral, because a risk neutral person would be indifferent between L and \$21.92 ($L \sim$ \$21.92); since Bob prefers \$21.92 to \$20 and has transitive preferences, if risk neutral he would prefer L to \$20.
However, Bob could also be risk loving: a risk-loving person prefers L to \$21.92 ($L \succ$ \$21.92) and we know that he prefers \$21.92 to \$20; thus, by transitivity, if risk loving, he would prefer L to \$20.
But Bob could also be risk averse: he could consistently prefer \$21.92 to L and L to \$20 (for example, he could consider L to be just as good as \$20.50). □

Solution to Exercise 5.4 Just like Exercise 5.2, this was a trick question! Here the basic outcomes are not sums of money but states of health. Since the described choice is not one between money lotteries, the definitions of risk aversion/neutrality/love are not applicable. □

Solution to Exercise 5.5 Since Ben prefers B to A, he must prefer D to C.
Proof. Let U be a von Neumann-Morgenstern utility function that represents Ben's preferences.
- Let $U(\$4,000) = a, U(\$3,000) = b$ and $U(\$0) = c$.
- Since Ben prefers more money to less, $a > b > c$.
- Then $\mathbb{E}[U(A)] = 0.8\,U(\$4,000) + 0.2\,U(\$0) = 0.8a + 0.2c$ and $\mathbb{E}[U(B)] = U(\$3,000) = b$.
- Since Ben prefers B to A, it must be that $b > 0.8a + 0.2c$.

Let us now compare C and D: $\mathbb{E}[U(C)] = 0.2\,a + 0.8c$ and $\mathbb{E}[U(D)] = 0.25b + 0.75c$.

- Since $b > 0.8a + 0.2c$, $\quad 0.25b > 0.25(0.8a + 0.2c) = 0.2a + 0.05c$ and thus, adding $0.75c$ to both sides, we get that $0.25b + 0.75c > 0.2a + 0.8c$, that is, $\mathbb{E}[U(D)] > \mathbb{E}[U(C)]$, so that $D \succ C$.

Note that the proof would have been somewhat easier if we had taken the normalized utility function, so that $a = 1$ and $c = 0$. □

Solution to Exercise 5.6 Define the function U as follows:
$U(x) = \frac{1}{a-c}V(x) - \frac{c}{a-c} = \frac{V(x)-c}{a-c}$ (note that, by hypothesis, $a > c$ and thus $\frac{1}{a-c} > 0$).
Then U represents the same preferences as V.
Then $U(o_2) = \frac{V(o_2)-c}{a-c} = \frac{a-c}{a-c} = 1, U(o_1) = \frac{V(o_1)-c}{a-c} = \frac{b-c}{a-c}$, and $U(o_3) = \frac{V(o_3)-c}{a-c} = \frac{c-c}{a-c} = 0$.
Note that, since $a > b > c$, $0 < \frac{b-c}{a-c} < 1$. □

Solution to Exercise 5.7 We can take the set of basic outcomes to be $\{\$3000, \$2000, \$1000, \$500\}$. Suppose that there is a von Neumann-Morgenstern utility function U that represents Jennifer's preferences. We can normalize it so that $U(\$3000) = 1$ and $U(\$500) = 0$.
- Since Jennifer is indifferent between L_1 and \$2000, $U(\$2000) = \frac{5}{6}$ (since the expected utility of L_1 is $\frac{5}{6}(1) + \frac{1}{6}(0) = \frac{5}{6}$).
- Since she is indifferent between L_2 and \$1000, $U(\$1000) = \frac{2}{3}$ (since the expected utility of L_2 is $\frac{2}{3}(1) + \frac{1}{3}(0) = \frac{2}{3}$).
Thus, $\mathbb{E}[U(L_3)] = \frac{1}{4}(1) + \frac{1}{4}\left(\frac{5}{6}\right) + \frac{1}{4}\left(\frac{2}{3}\right) + \frac{1}{4}(0) = \frac{5}{8}$ and $\mathbb{E}[U(L_4)] = \frac{1}{2}\left(\frac{5}{6}\right) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{3}{4}$.
Since $\frac{3}{4} > \frac{5}{8}$, Jennifer should prefer L_4 to L_3. Hence, she is not rational according to the theory of expected utility. □

Solution to Exercise 5.8 Normalize her utility function so that $U(o_1) = 1$ and $U(o_4) = 0$.
Since Rachel is indifferent between $\begin{pmatrix} o_2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} o_1 & o_4 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$, we have that $U(o_2) = \frac{4}{5}$.
Similarly, since she is indifferent between $\begin{pmatrix} o_3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} o_1 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $U(o_3) = \frac{1}{2}$.
Then the expected utility of $L_1 = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{8} & \frac{2}{8} & \frac{3}{8} & \frac{2}{8} \end{pmatrix}$ is $\frac{1}{8}(1) + \frac{2}{8}(\frac{4}{5}) + \frac{3}{8}(\frac{1}{2}) + \frac{2}{8}(0) = \frac{41}{80} = 0.5125$,
while the expected utility of $L_2 = \begin{pmatrix} o_1 & o_2 & o_3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$ is $\frac{1}{5}(1) + \frac{3}{5}(\frac{4}{5}) + \frac{1}{5}(\frac{1}{2}) =. \frac{39}{50} = 0.78$.
Hence, she prefers L_2 to L_1. □

Solution to Exercise 5.9

(a) The expected value of L_1 is $\frac{2}{10}(30) + \frac{1}{10}(28) + \frac{1}{10}(24) + \frac{2}{10}(18) + \frac{4}{10}(8) = 18$ and the expected value of L_2 is $\frac{1}{10}(30) + \frac{4}{10}(28) + \frac{5}{10}8 = 18.2$.

Hence, a risk-neutral person would prefer L_2 to L_1.

(b) The expected utility of L_1 is $\frac{1}{5}\ln(30) + \frac{1}{10}\ln(28) + \frac{1}{10}\ln(24) + \frac{1}{5}\ln(18) + \frac{2}{5}\ln(8) = 2.741$ while the expected utility of L_2 is $\frac{1}{10}\ln(30) + \frac{2}{5}\ln(28) + \frac{1}{2}\ln(8) = 2.713$.
Thus, Paul would choose L_1 (since he prefers L_1 to L_2). □

Solution to Exercise 5.10

(a) To normalize U first add 10 to each value and then divide by 180. Denote the normalization of U by $\overline{U}$.
Then

$$\overline{U}: \begin{array}{ccccc} o_1 & o_2 & o_3 & o_4 & o_5 \\ \frac{54}{180}=0.3 & \frac{180}{180}=1 & \frac{0}{180}=0 & \frac{36}{180}=0.2 & \frac{108}{180}=0.6 \end{array}$$

To normalize V first subtract 5 from each value and then divide by 90. Denote the normalization of V by $\overline{V}$.
Then

$$\overline{V}: \begin{array}{ccccc} o_1 & o_2 & o_3 & o_4 & o_5 \\ \frac{27}{90}=0.3 & \frac{90}{90}=1 & \frac{0}{90}=0 & \frac{18}{90}=0.2 & \frac{54}{90}=0.6 \end{array}$$

(b) The transformation is of the form $V(o) = aU(o) + b$. To find the values of a and b plug in two sets of values and solve the system of equations $\begin{cases} 44a+b=32 \\ 170a+b=95 \end{cases}$.
The solution is $a = \frac{1}{2}$, $b = 10$. Thus, $V(o) = \frac{1}{2}U(o) + 10$. □

Solution to Exercise 5.11

(a) Ann prefers L_3 to L_4 ($L_3 \succ_{Ann} L_4$). In fact, $\mathbb{E}[U_{Ann}(L_3)] = \sqrt{28} = 5.2915$ while $\mathbb{E}[U_{Ann}(L_4)] = \frac{1}{2}\sqrt{10} + \frac{1}{2}\sqrt{50} = 5.1167$.

(b) Bob prefers L_4 to L_3 ($L_4 \succ_{Bob} L_3$). In fact, $\mathbb{E}[U_{Bob}(L_3)] = 2(28) - \frac{28^4}{100^3} = 55.3853$ while $\mathbb{E}[U_{Bob}(L_4)] = \frac{1}{2}\left[2(10) - \frac{10^4}{100^3}\right] + \frac{1}{2}\left[2(50) - \frac{50^4}{100^3}\right] = 56.87$.

(c) The expected value of lottery L_4 is $\frac{1}{2}10 + \frac{1}{2}50 = 30$; thus, a risk-averse person would strictly prefer \$30 with certainty to the lottery L_4. We saw in part **(a)** that for Ann the expected utility of lottery L_4 is 5.1167; the utility of \$30 is $\sqrt{30} = 5.4772$. Thus, Ann would indeed choose \$30 for certain over the lottery L_4. We saw in part **(b)** that for Bob the expected utility of lottery L_4 is 56.87; the utility of \$30 is $2(30) - \frac{30^4}{100^3} = 59.19$. Thus, Bob would indeed choose \$30 for certain over the lottery L_4. □

Solution to Exercise 5.12 The simple lottery is $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{18}{240} & \frac{103}{240} & \frac{95}{240} & \frac{24}{240} \end{pmatrix}$. For example, the probability of o_2 is computed as follows: $\frac{1}{8}\left(\frac{1}{10}\right) + \frac{1}{4}(1) + \frac{1}{8}(0) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{103}{240}$. □

Solution to Exercise 5.13 Using the stated indifference, use lottery L to construct the compound lottery $\begin{pmatrix} \begin{pmatrix} o_2 & o_4 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} & \begin{pmatrix} o_3 & o_4 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} & o_3 & o_4 \\ \frac{1}{3} & \frac{2}{9} & \frac{1}{9} & \frac{1}{3} \end{pmatrix}$, whose corresponding simple lottery is $L' = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ 0 & \frac{1}{12} & \frac{11}{45} & \frac{121}{180} \end{pmatrix}$. Then, by the Independence Axiom, $L \sim L'$. $\square$

Solution to Exercise 5.14 Let W be your initial wealth. The basic outcomes are:

1. you do not pay any money, do not die and live to enjoy your wealth W (denote this outcome by A_0),
2. you pay \$$Y$, do not die and live to enjoy your remaining wealth $W - Y$ (call this outcome A_Y),
3. you pay \$$X$, do not die and live to enjoy your remaining wealth $W - X$ (call this outcome A_X),
4. you die (call this outcome D); this could happen because **(a)** you do not pay any money, roll the die and drink the poison or **(b)** you pay \$$Y$, roll the die and drink the poison; we assume that you are indifferent between these two outcomes.

Since, by hypothesis, $X > Y$, your ranking of these outcomes must be $A_0 \succ A_Y \succ A_X \succ D$. If you satisfy the von Neumann-Morgenstern axioms, then your preferences can be represented by a von Neumann-Morgenstern utility function U defined on the set of basic outcomes. We can normalize your utility function by setting $U(A_0) = 1$ and $U(D) = 0$. Furthermore, it must be that

$$U(A_Y) > U(A_X). \tag{5.6}$$

The maximum amount \$$P$ that you are willing to pay is that amount that makes you indifferent between **(1)** rolling the die with the initial number of poisoned glasses and **(2)** giving up \$$P$ and rolling the die with one less poisoned glass.

Thus – based on your answers – you are indifferent between the two lotteries

$$\begin{pmatrix} D & A_0 \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix} \text{ and } \begin{pmatrix} A_X \\ 1 \end{pmatrix}$$

and you are indifferent between the two lotteries:

$$\begin{pmatrix} D & A_0 \\ \frac{4}{6} & \frac{2}{6} \end{pmatrix} \text{ and } \begin{pmatrix} D & A_Y \\ \frac{3}{6} & \frac{3}{6} \end{pmatrix}.$$

Thus,

$$\underbrace{\tfrac{1}{6}U(D) + \tfrac{5}{6}U(A_0)}_{=\frac{1}{6}0+\frac{5}{6}1=\frac{5}{6}} = U(A_X) \text{ and } \underbrace{\tfrac{4}{6}U(D) + \tfrac{2}{6}U(A_0)}_{=\frac{4}{6}0+\frac{2}{6}1=\frac{2}{6}} = \underbrace{\tfrac{3}{6}U(D) + \tfrac{3}{6}U(A_Y)}_{=\frac{3}{6}0+\frac{3}{6}U(A_Y)}.$$

Hence, $U(A_X) = \frac{5}{6}$ and $U(A_Y) = \frac{2}{3} = \frac{4}{6}$, so that $U(A_X) > U(A_Y)$, contradicting (5.6). $\square$