11. Weak Sequential Equilibrium

11.1 Assessments and sequential rationality

At the end of Chapter 6 (Section 6.4) we showed that, although the notion of subgame-perfect equilibrium is a refinement of Nash equilibrium, it is not strong enough to eliminate all "unreasonable" Nash equilibria. One reason for this is that a subgame-perfect equilibrium σ allows a player's strategy to include a strictly dominated choice at an information set that is not reached by the play induced by σ. In order to eliminate such possibilities we need to define the notion of equilibrium for dynamic games in terms of a more complex object than merely a strategy profile. We need to add a description of what the players believe when it is their turn to move.

Definition 11.1.1 Given an extensive-form game G, an *assessment* for G is a pair (σ, μ), where σ is a profile of behavioral strategies and μ is a list of probability distributions, one for every information set, over the nodes in that information set. We call μ a *system of beliefs*.

The system of beliefs μ specifies, for every information set, the beliefs – about past moves – that the player who moves at that information set would have if told that her information set had been reached.

Consider, for example, the extensive-form game of Figure 11.1.

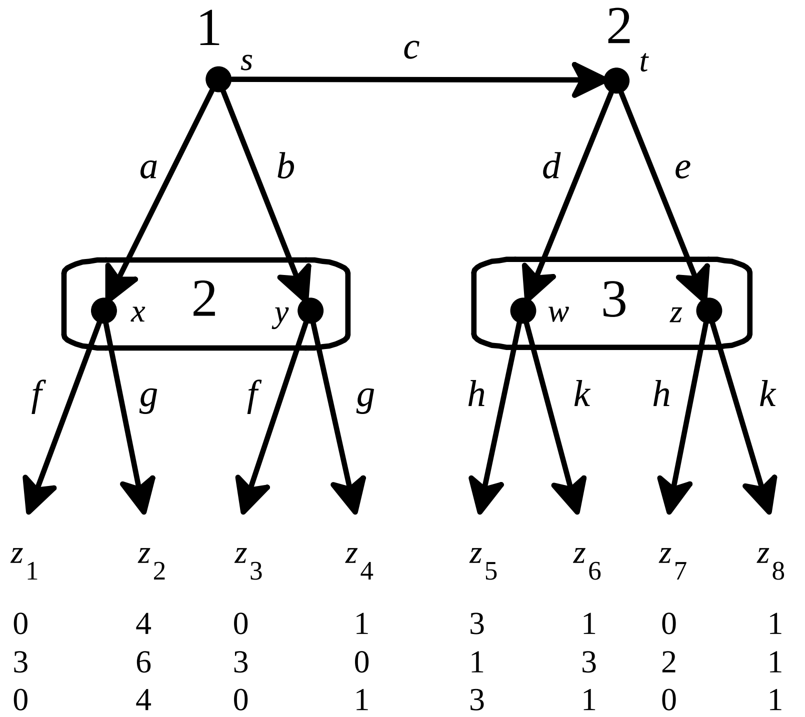

Figure 11.1: An extensive-form game with cardinal payoffs.

A possible assessment for this game is (σ, μ) with

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5} \end{array}\right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{array}\right).$$

Note that, typically, we will not bother to include in μ the trivial probability distributions over singleton information sets.[1]

The interpretation of this assessment is that

◇ Player 1 plans to play a with probability $\frac{1}{8}$, b with probability $\frac{3}{8}$ and c with probability $\frac{4}{8}$;

◇ Player 2 plans to play f if her information set $\{x,y\}$ is reached and to mix between d and e with probabilities $\frac{3}{4}$ and $\frac{1}{4}$, respectively, if her decision node t is reached;

◇ Player 3 plans to mix between h and k with probabilities $\frac{1}{5}$ and $\frac{4}{5}$, respectively, if his information set $\{w,z\}$ is reached;

◇ Player 2 – if informed that her information set $\{x,y\}$ had been reached – would attach probability $\frac{2}{3}$ to Player 1 having played a (node x) and probability $\frac{1}{3}$ to Player 1 having played b (node y);

[1] A complete specification of μ would be $\left(\begin{array}{c|c|cc|cc} s & t & x & y & w & z \\ 1 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$ where s is the decision node of Player 1 (the root) and t is the decision node of Player 2 following choice c.

◇ Player 3 – if informed that his information set $\{w,z\}$ had been reached – would attach probability $\frac{1}{2}$ to node w (that is, to the sequence of moves cd) and probability $\frac{1}{2}$ to node z (that is, to the sequence of moves ce).

In order for an assessment (σ,μ) to be considered "reasonable" we will impose two requirements:

1. The choices specified by σ should be optimal given the beliefs specified by μ. We call this requirement *sequential rationality*.
2. The beliefs specified by μ should be consistent with the strategy profile σ. We call this requirement *Bayesian updating*.

Before we give a precise definition of these concepts, we shall illustrate them with reference to the assessment considered in the game of Figure 11.1, namely

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5} \end{array}\right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{array}\right).$$

This assessment fails to satisfy sequential rationality, because, for example, at Player 3's information set $\{w,z\}$ the planned mixed strategy $\begin{pmatrix} h & k \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$ yields Player 3 – given her beliefs $\begin{pmatrix} w & z \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ – an expected payoff of $\frac{1}{2}\left[\frac{1}{5}(3)+\frac{4}{5}(1)\right]+\frac{1}{2}\left[\frac{1}{5}(0)+\frac{4}{5}(1)\right] = \frac{11}{10}$, while she could get a higher expected payoff, namely $\frac{1}{2}(3)+\frac{1}{2}(0) = \frac{3}{2} = \frac{15}{10}$, by playing h with probability 1.

This assessment also fails the rule for belief updating (Definition 9.4.1, Chapter 9); for example, given σ the prior probability that node x is reached is $P(x) = \frac{1}{8}$ (it is the probability with which Player 1 plays a) and the prior probability that node y is reached is $P(y) = \frac{3}{8}$ (the probability with which Player 1 plays b), so that updating on information $\{x,y\}$ one gets

$$P(x|\{x,y\}) = \frac{P(x)}{P(\{x,y\})} = \frac{P(x)}{P(x)+P(y)} = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{3}{8}} = \frac{1}{4}$$

and $P(y|\{x,y\}) = \frac{3}{4}$. Thus, in order to be consistent with the rule for belief updating, Player 2's beliefs should be

$$\begin{pmatrix} x & y \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

Now we can turn to the formal definitions. First we need to introduce some notation. If σ is a profile of behavior strategies and a is a choice of Player i (at some information set of Player i), we denote by $\sigma(a)$ the probability that σ_i (the strategy of Player i that is part of σ) assigns to a.
For example, for the game of Figure 11.1, if

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5} \end{array} \right)$$

then $\sigma(b) = \frac{3}{8}$, $\quad \sigma(g) = 0$, $\quad \sigma(d) = \frac{3}{4}$, etc.

Similarly, if μ is a system of beliefs and x is a decision node, then we denote by $\mu(x)$ the probability that the relevant part of μ assigns to x. For example, if

$$\mu = \left(\begin{array}{c|c|cc|cc} s & t & x & y & w & z \\ 1 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{array} \right)$$

or, written more succinctly,

$$\mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{array} \right)$$

then $\mu(s) = 1$, $\mu(y) = \frac{1}{3}$, $\mu(w) = \frac{1}{2}$, etc.

Recall that Z denotes the set of terminal nodes and, for every Player i, $\pi_i : Z \to \mathbb{R}$ is the payoff function of Player i.

- Given a decision node x, let $Z(x) \subseteq Z$ be the set of terminal nodes that can be reached starting from x. For example, in the game of Figure 11.1, $Z(t) = \{z_5, z_6, z_7, z_8\}$.
- Given a behavior strategy profile σ and a decision node x, let $\mathbb{P}_{x,\sigma}$ be the probability distribution over $Z(x)$ induced by σ, that is, if $z \in Z(x)$ and $\langle a_1, ..., a_m \rangle$ is the sequence of choices that leads from x to z then $\mathbb{P}_{x,\sigma}(z)$ is the product of the probabilities of those choices: $\mathbb{P}_{x,\sigma}(z) = \sigma(a_1) \times \sigma(a_2) \times \ldots \times \sigma(a_m)$.

For example, in the game of Figure 11.1, if σ is the strategy profile

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5} \end{array} \right)$$

and t is Player 2's decision node after choice c of Player 1, then
$\mathbb{P}_{t,\sigma}(z_5) = \sigma(d)\,\sigma(h) = \frac{3}{4}\left(\frac{1}{5}\right) = \frac{3}{20}$.

If H is an information set of Player i we denoted by $\pi_i(H|\sigma,\mu)$ the expected payoff of Player i starting from information set H, given the beliefs specified by μ at H and given the choices prescribed by σ at H and at the information sets that come after H, that is,

$$\pi_i(H|\sigma,\mu) = \sum_{x\in H}\left[\mu(x)\left(\sum_{z\in Z(x)}\mathbb{P}_{x,\sigma}(z)\,\pi_i(z)\right)\right].$$

For example, in the game of Figure 11.1, if

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5}\end{array}\right) \quad\text{and}\quad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2}\end{array}\right),$$

then (as we computed above)

$$\begin{aligned}
&\pi_3\Big(\{w,z\}|\sigma,\mu\Big) = \\
&= \mu(w)\Big(\mathbb{P}_{w,\sigma}(z_5)\,\pi_3(z_5) + \mathbb{P}_{w,\sigma}(z_6)\,\pi_3(z_6)\Big) + \mu(z)\Big(\mathbb{P}_{z,\sigma}(z_7)\,\pi_3(z_7) + \mathbb{P}_{z,\sigma}(z_8)\,\pi_3(z_8)\Big) \\
&= \tfrac{1}{2}\left[\tfrac{1}{5}(3) + \tfrac{4}{5}(1)\right] + \tfrac{1}{2}\left[\tfrac{1}{5}(0) + \tfrac{4}{5}(1)\right] = \tfrac{11}{10}.
\end{aligned}$$

Recall that if σ is a strategy profile and i is a player, then σ_{-i} denotes the profile of strategies of the players other than i and we can use (σ_i,σ_{-i}) as an alternative way of denoting σ; furthermore, if τ_i is a strategy of Player i, we denote by (τ_i,σ_{-i}) the strategy profile obtained from σ by replacing σ_i with τ_i (and leaving everything else unchanged).

Definition 11.1.2 Fix an extensive-form game and an *assessment* (σ,μ). We say that Player i's behavior strategy σ_i is *sequentially rational* if, for every information set H of Player i, $\pi_i(H|(\sigma_i,\sigma_{-i}),\mu) \geq \pi_i(H|(\tau_i,\sigma_{-i}),\mu)$, for every behavior strategy τ_i of Player i. We say that σ is *sequentially rational* if, for every Player i, σ_i is sequentially rational.

Note that for Player i's strategy σ_i to be sequentially rational it is not sufficient (although it is necessary) that at every information set H of Player i the choice(s) at H prescribed by σ_i be optimal (given the choices of the other players specified by σ_{-i}): we need to check if Player i could improve her payoff by changing her choice(s) not only at H *but also at information sets of hers that follow H*. To see this, consider the game as shown in Figure 11.2.

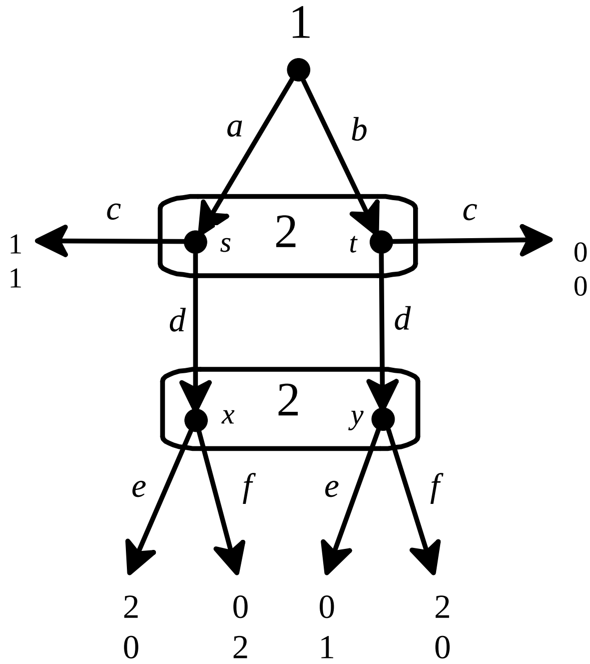

Figure 11.2: A strategic-form game with cardinal payoffs.

Let (σ,μ) be the assessment where σ is the pure-strategy profile $(a,(c,e))$ and

$$\mu = \left(\begin{array}{cc|cc} s & t & x & y \\ 1 & 0 & 0 & 1 \end{array}\right).$$

For Player 2, e is rational at information set $\{x,y\}$ because – given her belief that she is making her choice at node y – e gives her a payoff of 1 while f would give her a payoff of 0;
furthermore, at information set $\{s,t\}$ – given her belief that she is making her choice at node s *and given her future choice of* e at $\{x,y\}$ – choice c is better than choice d because the former gives her a payoff of 1 while the latter gives her a payoff of 0.

However, the *strategy* (c,e) (while sequentially rational at $\{x,y\}$) is not sequentially rational at $\{s,t\}$ because – given her belief that she is making her choice at node s – with (c,e) she gets a payoff of 1 but if she switched to (d,f) she would get a payoff of 2; in other words, Player 2 can improve her payoff by changing her choice at $\{s,t\}$ from c to d *and also* her future planned choice at $\{x,y\}$ from e to f.[2]

Note that the pure-strategy profile $(a,(c,e))$ is not a Nash equilibrium: for Player 2 the unique best reply to a is (d,f).

[2]It is possible to impose restrictions on the system of beliefs μ such that sequential rationality as defined in Definition 11.1.2 is equivalent to the weaker condition that, at every information set H, the corresponding player cannot increase her payoff by changing her choice(s) at H only. We will not discuss this condition here. The interested reader is referred to Hendon et al (1996) and Perea (2002).

An example of a sequentially rational assessment for the game shown in Figure 11.3 is

$$\sigma = \begin{pmatrix} L & M & R & A & B & c & d \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} x & y & u & v & w \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

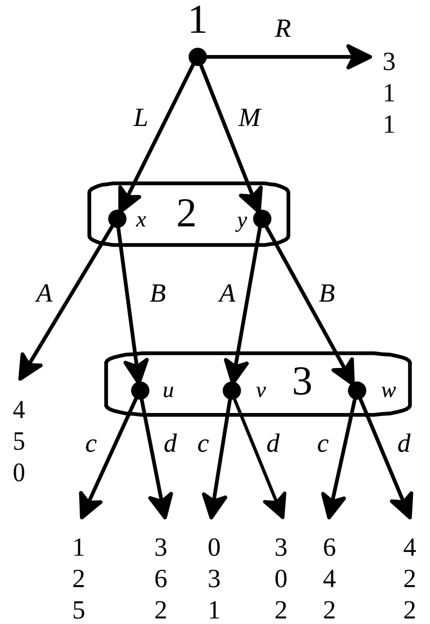

Figure 11.3: The assessment $\sigma = (R, \left(\frac{1}{2}A, \frac{1}{2}B\right), c)$, $\mu = \left(\left(\frac{1}{4}x, \frac{3}{4}y\right), \left(\frac{1}{5}u, \frac{3}{5}v, \frac{1}{5}w\right)\right)$ is sequentially rational.

Let us first verify sequential rationality of Player 3's strategy.

At her information set $\{u, v, w\}$, given her beliefs,

- c gives a payoff of $\frac{1}{5}(5) + \frac{3}{5}(1) + \frac{1}{5}(2) = 2$,
- while d gives a payoff of $\frac{1}{5}(2) + \frac{3}{5}(2) + \frac{1}{5}(2) = 2$.
- Thus c is optimal (as would be d and any randomization over c and d).

Now consider Player 2:

At his information set $\{x, y\}$, given his beliefs and given the strategy of Player 3,

- A gives a payoff of $\frac{1}{4}(5) + \frac{3}{4}(3) = 3.5$
- and B gives a payoff of $\frac{1}{4}(2) + \frac{3}{4}(4) = 3.5$;
- thus any mixture of A and B is optimal, in particular, the mixture $\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is optimal.

Finally, at the root, R gives Player 1 a payoff of 3, L a payoff of $\frac{1}{2}(4) + \frac{1}{2}(1) = 2.5$ and M a payoff of $\frac{1}{2}(0) + \frac{1}{2}(6) = 3$; thus R is optimal (as would be any mixture over M and R).

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 11.4.1 at the end of this chapter.

11.2 Bayesian updating at reached information sets

The second requirement for an assessment to be "reasonable" is that the beliefs encoded in μ should be consistent with the behavior postulated by σ in the sense that the rule for belief updating (Definition 9.4.1, Chapter 9), should be used to form those beliefs, whenever it is applicable. We shall call this rule *Bayesian updating*.

Let x be a node that belongs to information set H and let $\mathbb{P}_{root,\sigma}(x)$ be the probability that node x is reached (from the root of the tree) if σ is implemented.[3]

Let $\mathbb{P}_{root,\sigma}(H) = \sum_{y \in H} \mathbb{P}_{root,\sigma}(y)$ be the probability that information set H is reached (that is, the probability that some node in H is reached).

Then Bayesian updating requires that the probability that is assigned to x – given the information that H has been reached – be given by the conditional probability $\mathbb{P}_{root,\sigma}(x|H) = \frac{\mathbb{P}_{root,\sigma}(x)}{\mathbb{P}_{root,\sigma}(H)}$;

of course, this conditional probability is well defined if and only if $\mathbb{P}_{root,\sigma}(H) > 0$.

Definition 11.2.1 Consider an extensive-form game and a behavioral strategy profile σ. We say that an information set H is *reachable by* σ if $\mathbb{P}_{root,\sigma}(H) > 0$. That is, H is *reachable by* σ if at least one node in H is reached with positive probability when the game is played according to σ.

For example, in the game-frame of Figure 11.4 (which partially reproduces of Figure 11.1 by omitting the payoffs), if

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} & 1 & 0 \end{array} \right)$$

then $\{x, y\}$ is reachable (with probability 1) by σ, while $\{w, z\}$ is not.[4]

Definition 11.2.2 Given an extensive-form game and an assessment (σ, μ), we say that (σ, μ) satisfies *Bayesian updating at reachable information sets* if, for every information set H and every node $x \in H$, if H is reachable by σ (that is, if $\mathbb{P}_{root,\sigma}(H) > 0$) then $\mu(x) = \frac{\mathbb{P}_{root,\sigma}(x)}{\mathbb{P}_{root,\sigma}(H)}$.

For example, in the game-frame of Figure 11.4, the assessment

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{9} & \frac{5}{9} & \frac{3}{9} & 0 & 1 & \frac{3}{4} & \frac{1}{4} & 1 & 0 \end{array} \right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{1}{6} & \frac{5}{6} & \frac{3}{4} & \frac{1}{4} \end{array} \right)$$

satisfies Bayesian updating at reachable information sets.

[3] That is, if $\langle a_1, \ldots, a_m \rangle$ is the sequence of choices that leads from the root to x then $\mathbb{P}_{root,\sigma}(x) = \sigma(a_1) \times \cdots \times \sigma(a_m)$.

[4] $\mathbb{P}_{root,\sigma}(x) = \frac{1}{3}$, $\mathbb{P}_{root,\sigma}(y) = \frac{2}{3}$ and thus $\mathbb{P}_{root,\sigma}(\{x,y\}) = \frac{1}{3} + \frac{2}{3} = 1$; on the other hand, $\mathbb{P}_{root,\sigma}(w) = \sigma(c)\,\sigma(d) = 0\left(\frac{3}{4}\right) = 0$ and $\mathbb{P}_{root,\sigma}(z) = \sigma(c)\,\sigma(e) = 0\left(\frac{1}{4}\right) = 0$, so that $\mathbb{P}_{root,\sigma}(\{w,z\}) = 0$.

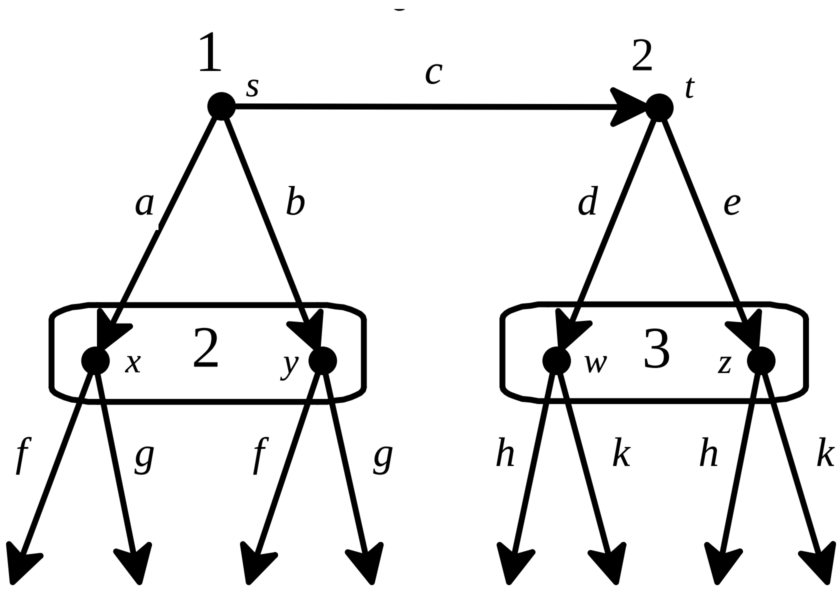

Figure 11.4: A game-frame in extensive form.

In fact, we have that

$\mathbb{P}_{root,\sigma}(x) = \frac{1}{9}, \qquad \mathbb{P}_{root,\sigma}(y) = \frac{5}{9}, \qquad \mathbb{P}_{root,\sigma}(\{x,y\}) = \frac{1}{9} + \frac{5}{9} = \frac{6}{9},$

$\mathbb{P}_{root,\sigma}(w) = \frac{3}{9}\left(\frac{3}{4}\right) = \frac{9}{36}, \qquad \mathbb{P}_{root,\sigma}(z) = \frac{3}{9}\left(\frac{1}{4}\right) = \frac{3}{36}$

and $\mathbb{P}_{root,\sigma}(\{w,x\}) = \frac{9}{36} + \frac{3}{36} = \frac{12}{36}.$

Thus

$$\frac{\mathbb{P}_{root,\sigma}(x)}{\mathbb{P}_{root,\sigma}(\{x,y\})} = \frac{\frac{1}{9}}{\frac{6}{9}} = \frac{1}{6} = \mu(x), \qquad \frac{\mathbb{P}_{root,\sigma}(y)}{\mathbb{P}_{root,\sigma}(\{x,y\})} = \frac{\frac{5}{9}}{\frac{6}{9}} = \frac{5}{6} = \mu(y),$$

$$\frac{\mathbb{P}_{root,\sigma}(w)}{\mathbb{P}_{root,\sigma}(\{w,z\})} = \frac{\frac{9}{36}}{\frac{12}{36}} = \frac{3}{4} = \mu(w) \quad \text{and} \quad \frac{\mathbb{P}_{root,\sigma}(z)}{\mathbb{P}_{root,\sigma}(\{w,z\})} = \frac{\frac{3}{36}}{\frac{12}{36}} = \frac{1}{4} = \mu(z).$$

R Note that the condition "Bayesian updating at reachable information sets" is trivially satisfied at every information set H that is *not* reachable by σ that is, at every information set H such that $\mathbb{P}_{root,\sigma}(H) = 0$.[5]

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 11.4.2 at the end of this chapter.

[5]In logic a proposition of the form "if A then B" (A is called the *antecedent* and B is called the *consequent*) is false only when A is true and B is false. Thus, in particular, the proposition "if A then B" is true whenever A is false (whatever the truth value of B). In our case the antecedent is "$\mathbb{P}_{root,\sigma}(H) > 0$".

11.3 A first attempt: Weak sequential equilibrium

Our objective is to find a refinement of subgame-perfect equilibrium that rules out "unreasonable" equilibria. Moving from strategy profiles to assessments allowed us to judge the rationality of a choice at an information set independently of whether that information set is reached by the strategy profile under consideration: the requirement of sequential rationality rules out choices that are strictly dominated at an information set. As a first attempt we define a notion of equilibrium based on the two requirements of sequential rationality and Bayesian updating at reached information sets.

Definition 11.3.1 A *weak sequential equilibrium* is an assessment (σ, μ) which satisfies two requirements: (1) sequential rationality and (2) Bayesian updating at reached information sets.

Before we discuss the properties of weak sequential equilibria we illustrate the type of reasoning that one needs to go through in order to find weak sequential equilibria. Consider first the game of Figure 11.5.

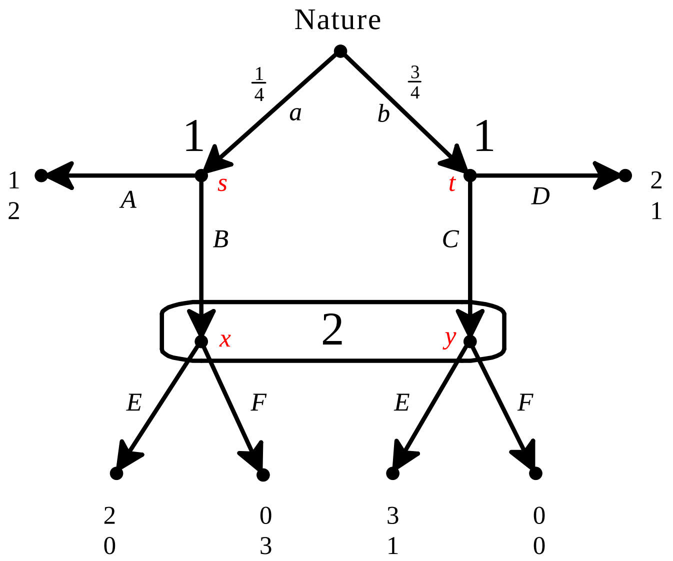

Figure 11.5: An extensive-form game with cardinal payoffs.

Let us see if there is a weak sequential equilibrium (σ, μ) where Player 1's strategy in σ is a pure strategy. The set of pure strategies of Player 1 is $S_1 = \{(A,C),(B,D),(B,C),(A,D)\}$. The strategy of Player 1 determines the beliefs of Player 2 at her information set $\{x,y\}$. Let us consider the four possibilities.

- If Player 1's strategy is (A,C), then Player 2's information set $\{x,y\}$ is reached with positive probability and the only beliefs that are consistent with Bayesian updating are $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, so that – by sequential rationality – Player 2 must choose E.

  However, if Player 2's strategy is E then at node s it is not sequentially rational for Player 1 to choose A. Thus there is no weak sequential equilibrium where Player 1's strategy is (A,C).

- If Player 1's strategy is (B,D), then Player 2's information set $\{x,y\}$ is reached with positive probability and the only beliefs that are consistent with Bayesian updating are $\begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}$, so that – by sequential rationality – Player 2 must choose F.

  However, if Player 2's strategy is F then at node s it is not sequentially rational for Player 1 to choose B. Thus there is no weak sequential equilibrium where Player 1's strategy is (B,D).

- If Player 1's strategy is (B,C), then Player 2's information set $\{x,y\}$ is reached (with probability 1) and the only beliefs that are consistent with Bayesian updating are $\begin{pmatrix} x & y \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.
  - Given these beliefs, Player 2's payoff from playing E is $\frac{1}{4}(0)+\frac{3}{4}(1)=\frac{3}{4}$ and her payoff from playing F is $\frac{1}{4}(3)+\frac{3}{4}(0)=\frac{3}{4}$.
  - Thus any mixed strategy is sequentially rational for Player 2, that is, for any $p\in[0,1]$, $\begin{pmatrix} E & F \\ p & 1-p \end{pmatrix}$ is sequentially rational.
  - At node s choice B is sequentially rational for Player 1 if and only if the expected payoff from playing B is at least 1 (which is the payoff from playing A): $2p+0(1-p)\geq 1$, that is, $p\geq\frac{1}{2}$.
  - At node t choice C is sequentially rational for Player 1 if and only if the expected payoff from playing C is at least 2 (which is the payoff from playing D): $3p+0(1-p)\geq 2$, that is, $p\geq\frac{2}{3}$.
  - Hence, if $p\geq\frac{2}{3}$ then both B and C are sequentially rational. Thus we have an infinite number of weak sequential equilibria at which Player 1's strategy is (B,C): for every $p\in\left[\frac{2}{3},1\right]$, (σ,μ) is a weak sequential equilibrium, where

$$\sigma=\left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 0 & 1 & 1 & 0 & p & 1-p \end{array}\right) \qquad \text{and} \qquad \mu=\begin{pmatrix} x & y \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

- If Player 1's strategy is (A,D), then Player 2's information set $\{x,y\}$ is not reached and thus, according to the notion of weak sequential equilibrium, any beliefs are allowed there. Let $\begin{pmatrix} E & F \\ p & 1-p \end{pmatrix}$ be Player 2's strategy. From previous calculations we have that if $p\leq\frac{1}{2}$ then A is sequentially rational at node s and D is sequentially rational at node t.

  One possibility is to set p = 0; this means that Player 2 chooses the pure strategy F and this is sequentially rational if and only if her beliefs are $\begin{pmatrix} x & y \\ q & 1-q \end{pmatrix}$ with

$$\underbrace{3q+0(1-q)}_{\text{payoff from } F} \geq \underbrace{0q+1(1-q)}_{\text{payoff from } E}, \qquad \text{that is,} \qquad q\geq\tfrac{1}{4}.$$

  Thus we have an infinite number of weak sequential equilibria: for every $q\in\left[\frac{1}{4},1\right]$, (σ,μ) is a weak sequential equilibrium, where

$$\sigma=\left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \qquad \text{and} \qquad \mu=\begin{pmatrix} x & y \\ q & 1-q \end{pmatrix}.$$

Next we consider the more complex game shown in Figure 11.6.

For simplicity, let us limit the search to pure-strategy weak sequential equilibria. How should we proceed? The first step is to see if the game itself can be simplified by checking if there are any information sets where there is a strictly dominant choice: if there is such a choice then, no matter what beliefs the relevant player has at that information set, sequential rationality requires that choice to be selected.
In the game of Figure 11.6 there is indeed such an information set, namely information set $\{w,z\}$ of Player 2: here L is strictly better than R for Player 2 at both nodes, that is, R is strictly dominated by L.

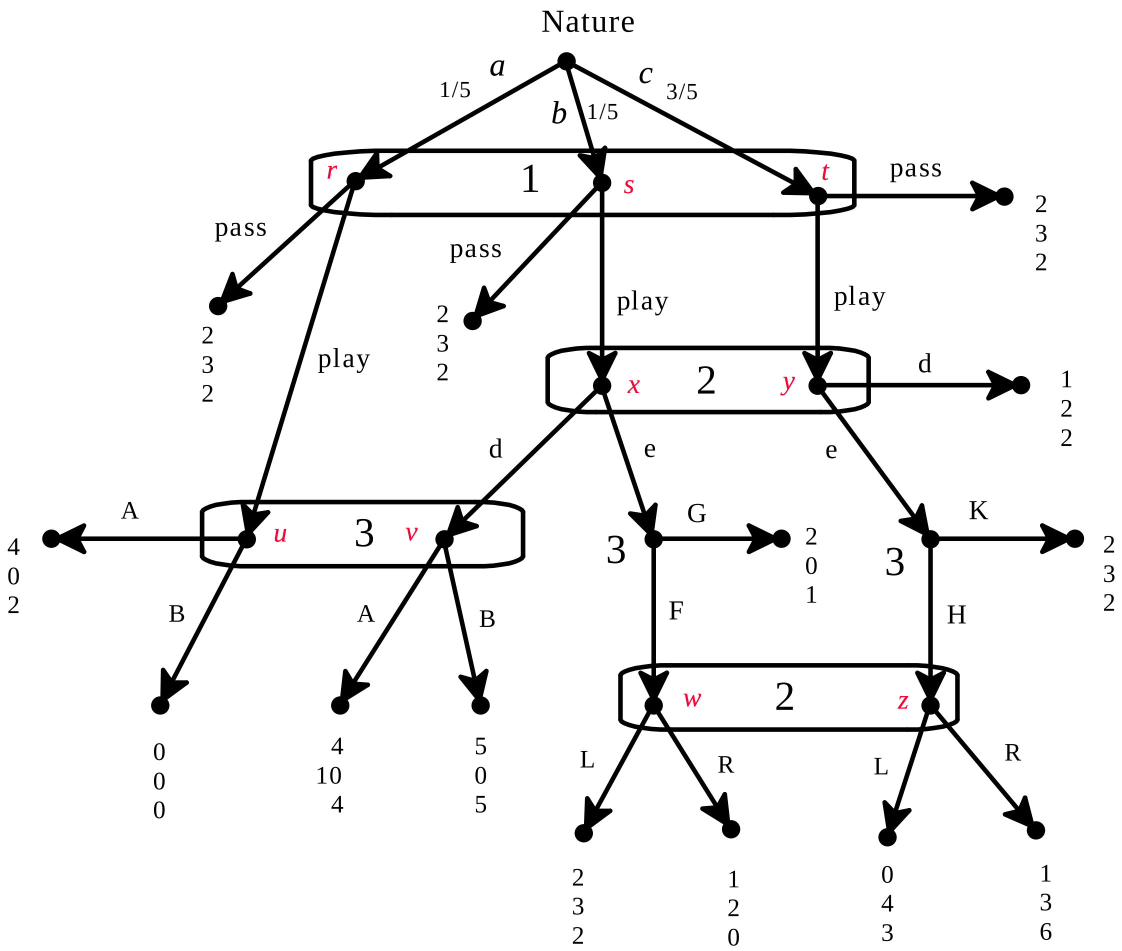

Figure 11.6: An extensive-form game with cardinal payoffs.

Thus we can simplify the game by

1. removing the information set $\{w, z\}$,
2. converting nodes w and z into terminal nodes, and
3. assigning to these newly created terminal nodes the payoffs associated with choice L.

The simplified game is shown in Figure 11.7.

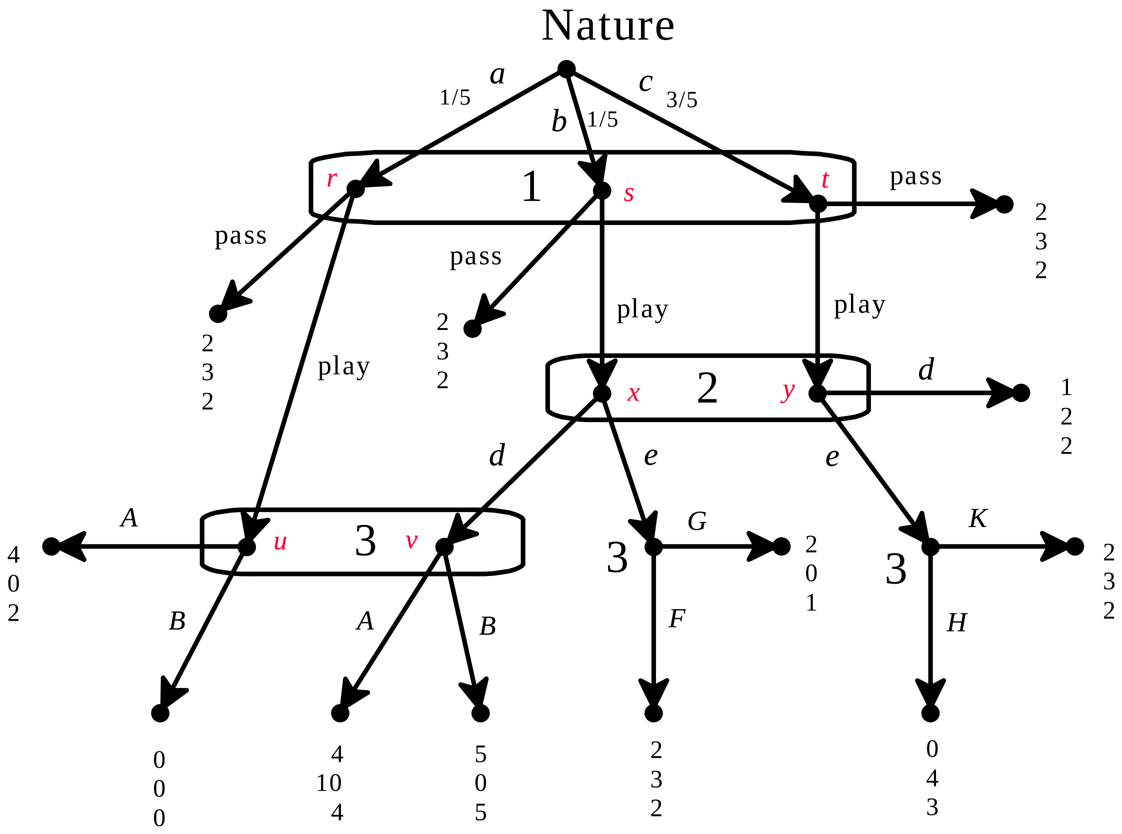

Figure 11.7: The game of Figure 11.6 simplified.

Applying the same reasoning to the simplified game of Figure 11.7, we can delete the two singleton decision nodes of Player 3 and replace the one on the left with the payoff associated with choice F and the one on the right with the payoff associated with choice H, thus obtaining the further simplification shown in Figure 11.8.

In this game there are no more information sets with strictly dominant choices. Thus we have to proceed by trial and error.
Note first that, at any weak sequential equilibrium, by Bayesian updating Player 1's beliefs must be $\begin{pmatrix} r & s & t \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \end{pmatrix}$.

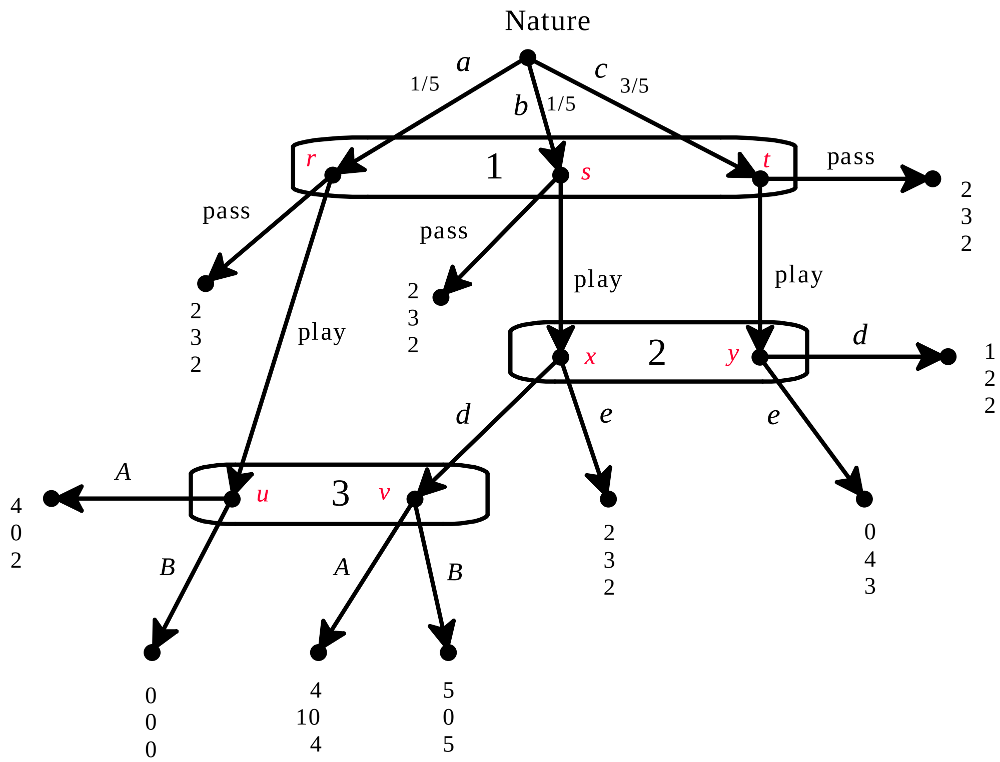

Figure 11.8: The game of Figure 11.7 simplified.

Let us see if in the simplified game of Figure 11.8 there is a pure-strategy weak sequential equilibrium where Player 1's strategy is "play".
When Player 1 chooses "play" then, by Bayesian updating, Player 2's beliefs must be $\begin{pmatrix} x & y \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

- Let us first try the hypothesis that there is a weak sequential equilibrium where (Player 1's strategy is "play" and) Player 2's strategy is e. Then, by Bayesian updating, Player 3's beliefs must be $\begin{pmatrix} u & v \\ 1 & 0 \end{pmatrix}$, making A the only sequentially rational choice at information set $\{u, v\}$.
  However, if Player 3's strategy is A then Player 2, at her information set $\{x, y\}$, gets a payoff of $\frac{1}{4}(10) + \frac{3}{4}(2) = \frac{16}{4}$ from playing d and a payoff of $\frac{1}{4}(3) + \frac{3}{4}(4) = \frac{15}{4}$ from playing e.
  Thus e is not sequentially rational and we have reached a contradiction.

- Let us now try the hypothesis that there is a weak sequential equilibrium where (Player 1's strategy is "play" and) Player 2's strategy is d.

  Then, by Bayesian updating, Player 3's beliefs must be $\begin{pmatrix} u & v \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$;

  hence Player 3's expected payoff from playing A is $\frac{1}{2}(2)+\frac{1}{2}(4)=3$

  and his expected payoff from playing B is $\frac{1}{2}(0)+\frac{1}{2}(5)=2.5$.

  Thus A is the only sequentially rational choice at information set $\{u,v\}$.

  Hence, by the previous calculations for Player 2, d is indeed sequentially rational for Player 2.

  It only remains to check if Player 1 indeed wants to choose "play".

  Given the strategies d and A of Players 2 and 3, respectively, Player 1 gets a payoff of 2 from "pass" and a payoff of $\frac{1}{5}(4)+\frac{1}{5}(4)+\frac{3}{5}(1)=\frac{11}{5}>2$ from "play".

  Thus "play" is indeed sequentially rational.

Thus we have found a pure-strategy weak sequential equilibrium of the game of Figure 11.8, namely

$$\sigma = \left(\begin{array}{cc|cc|cc} pass & play & d & e & A & B \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right) \quad \mu = \left(\begin{array}{ccc|cc|cc} r & s & t & x & y & u & v \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & \frac{1}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

This equilibrium can be extended to a weak sequential equilibrium of the original game of Figure 11.6 by adding the choices that led to the simplified game of Figure 11.8 and arbitrary beliefs at information set $\{w,z\}$: for any $p\in[0,1]$,

$$\sigma = \left(\begin{array}{cc|cc|cc|cc|cc|cc} pass & play & d & e & A & B & F & G & H & K & L & R \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\text{and } \mu = \left(\begin{array}{ccc|cc|cc|cc} r & s & t & x & y & u & v & w & z \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & \frac{1}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & p & 1-p \end{array}\right),$$

Are there any other pure-strategy weak sequential equilibria? This question is addressed in Exercise 11.7.

We now explore the relationship between the notion of weak sequential equilibrium and other equilibrium concepts. The proof of the following theorem is omitted.

Theorem 11.3.1 Given an extensive-form game with cardinal payoffs G, if (σ,μ) is a weak sequential equilibrium of G then σ is a Nash equilibrium of G.

In general, not every Nash equilibrium can be part of a weak sequential equilibrium. To see this, consider the extensive-form game of Figure 11.9.

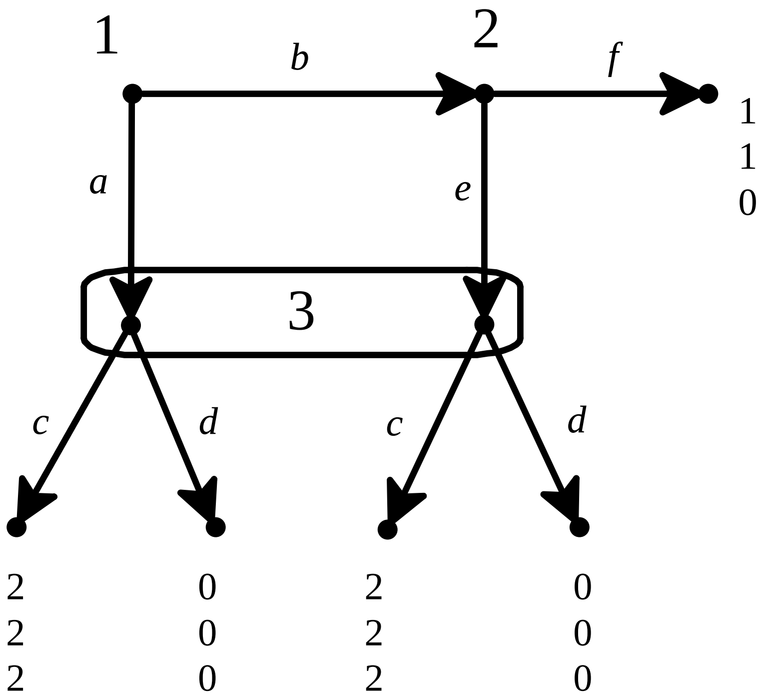

Figure 11.9: A game with a Nash equilibrium which is not part of any weak sequential equilibria.

The pure-strategy profile (b, f, d) is a Nash equilibrium, but it cannot be part of any weak sequential equilibrium, because – no matter what beliefs Player 3 has at her information set – choice d is not sequentially rational (it is strictly dominated by c at both nodes in that information set).

Thus weak sequential equilibrium is a strict refinement of Nash equilibrium. Does it also refine the notion of subgame-perfect equilibrium? Unfortunately, the answer is negative: it is possible for (σ, μ) to be a weak sequential equilibrium without σ being a subgame-perfect equilibrium.

To see this, consider the game of Figure 11.10 (which reproduces Figure 11.1).

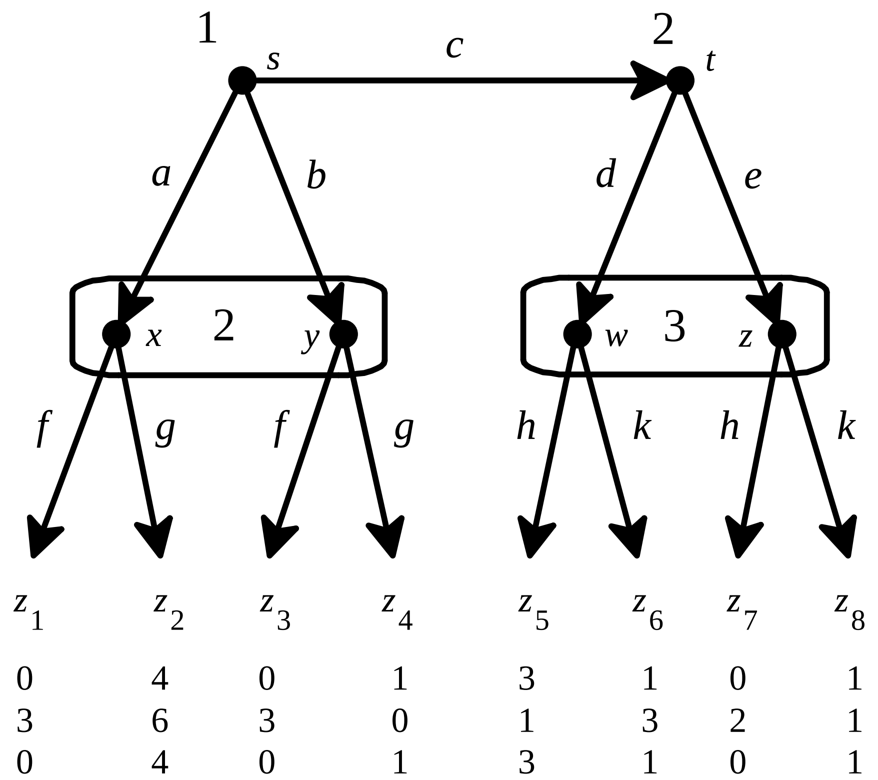

Figure 11.10: A game with a weak sequential equilibrium whose strategy profile is not a subgame-perfect equilibrium.

The assessment

$$\sigma = (b, (f, e), h) \qquad \text{and} \qquad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ 0 & 1 & 1 & 0 \end{array} \right)$$

is a weak sequential equilibrium (the reader should verify this; in particular, note that information set $\{w, z\}$ is not reached and thus Bayesian updating allows for arbitrary beliefs at that information set).
However, $(b, (f, e), h)$ is not a subgame-perfect equilibrium, because the restriction of this strategy profile to the proper subgame that starts at note t of Player 2, namely (e, h), is not a Nash equilibrium of that subgame: h is not a best reply to e.

The relationship between the three notions of Nash equilibrium, subgame-perfect equilibrium and weak sequential equilibrium is illustrated in the Venn diagram of Figure 11.11.

- Every subgame-perfect equilibrium is a Nash equilibrium. However, there are games in which there is a subgame-perfect equilibrium that is not part of any weak sequential equilibrium. For example, in the game of Figure 11.9, (b, f, d) is a Nash equilibrium which is also subgame-perfect (because there are no proper subgames); however, choice d is strictly dominated and thus is not sequentially rational for Player 3, no matter what beliefs he has at his information set. Thus (b, f, d) cannot be part of a weak sequential equilibrium.
- Every weak sequential equilibrium is a Nash equilibrium. However, as shown in the example of Figure 11.10, there are games in which there is a weak sequential equilibrium whose strategy is not a subgame-perfect equilibrium.

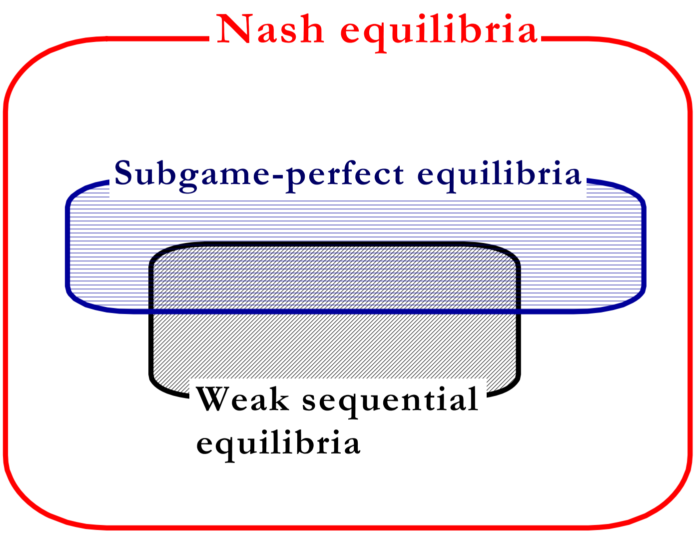

Figure 11.11: The relationship between Nash, subgame-perfect and weak sequential equilibrium.

In the next chapter we define the notion of sequential equilibrium and state two results: (1) a sequential equilibrium is a weak sequential equilibrium and (2) every finite extensive-form game with cardinal payoffs has at least one sequential equilibrium. As a corollary we obtain the following result.

Theorem 11.3.2 Every finite extensive-form game with cardinal payoffs has at least one weak sequential equilibrium (possibly in mixed strategies).

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 11.4.3 at the end of this chapter.

11.4 Exercises

11.4.1 Exercises for Section 11.1: Assessments and sequential rationality

The answers to the following exercises are in Section 11.5 at the end of this chapter.

Exercise 11.1

For the game of Figure 11.12 (which reproduces Figure 11.1), check whether the following assessment is sequentially rational.

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & 0 & 1 & 1 & 0 \end{array} \right),$$

$$\mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{2} \end{array} \right)$$

■

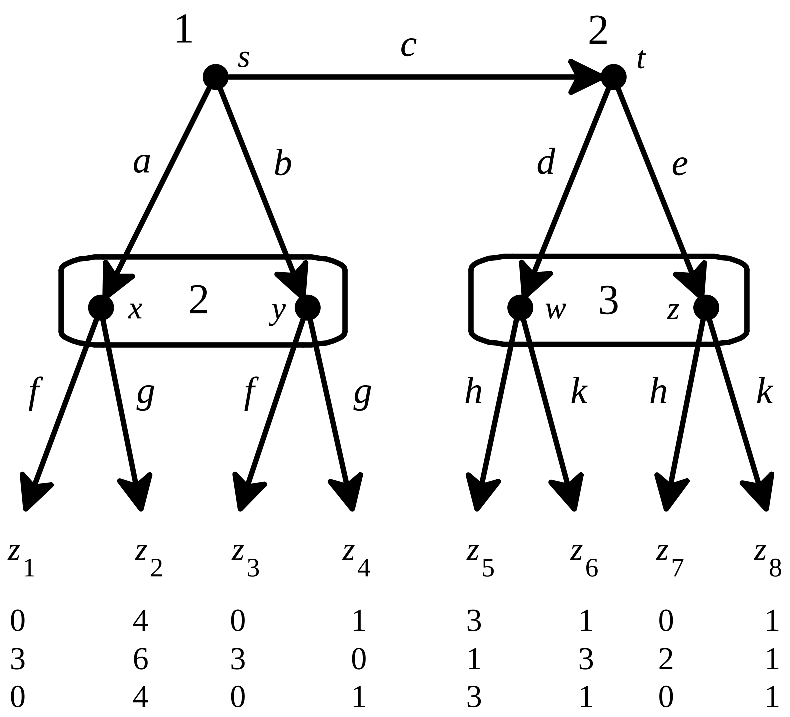

Figure 11.12: Copy of Figure 11.1.

Exercise 11.2

Consider the game of Figure 11.13, obtained from the game of Figure 11.12 by replacing Player 3 with Player 1 at information set $\{w,z\}$
$\big($note that in the game of Figure 11.12 the payoffs of Players 1 and 3 are identical$\big)$.
Is the following assessment sequentially rational?

$$\sigma = \begin{pmatrix} a & b & c & | & f & g & | & d & e & | & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & | & 1 & 0 & | & 0 & 1 & | & 1 & 0 \end{pmatrix},$$

$$\mu = \begin{pmatrix} x & y & | & w & z \\ \frac{1}{3} & \frac{2}{3} & | & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

■

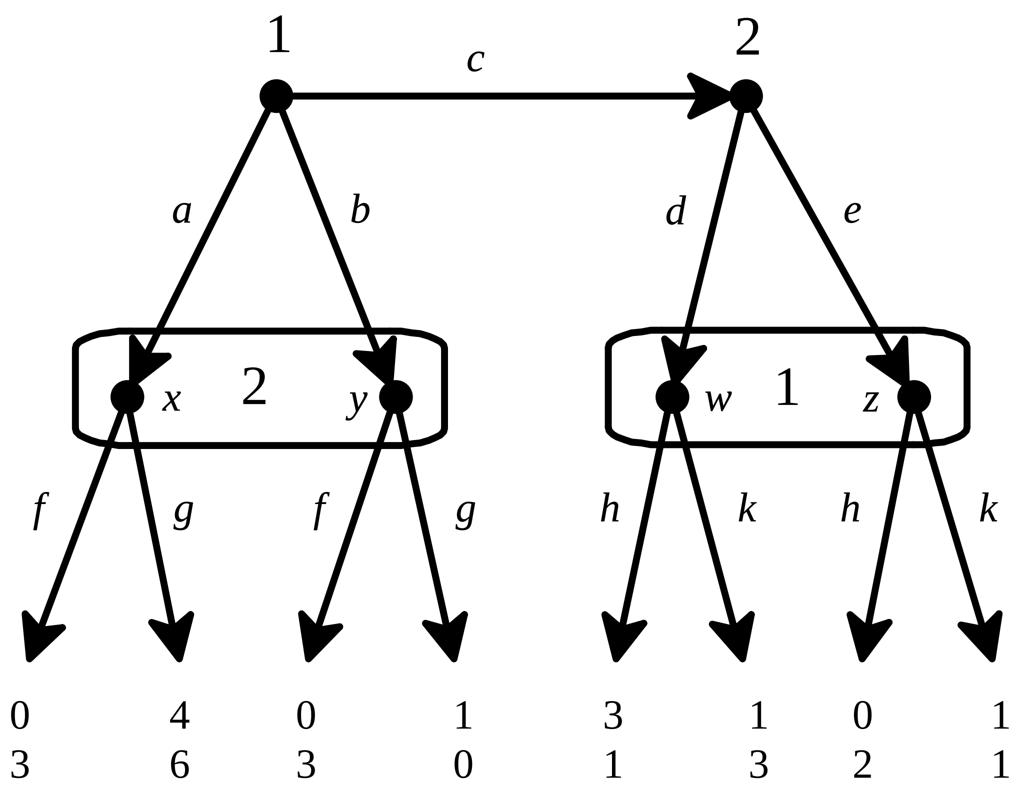

Figure 11.13: A two-player version of the game of Figure 11.12.

Exercise 11.3

Consider the game shown in Figure 11.14. Is the assessment

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & r & c & d & e & f \\ 0 & 0 & 1 & \frac{2}{5} & \frac{3}{5} & \frac{1}{3} & \frac{2}{3} \end{array}\right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} s & t & x & y \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{array}\right)$$

sequentially rational? ■

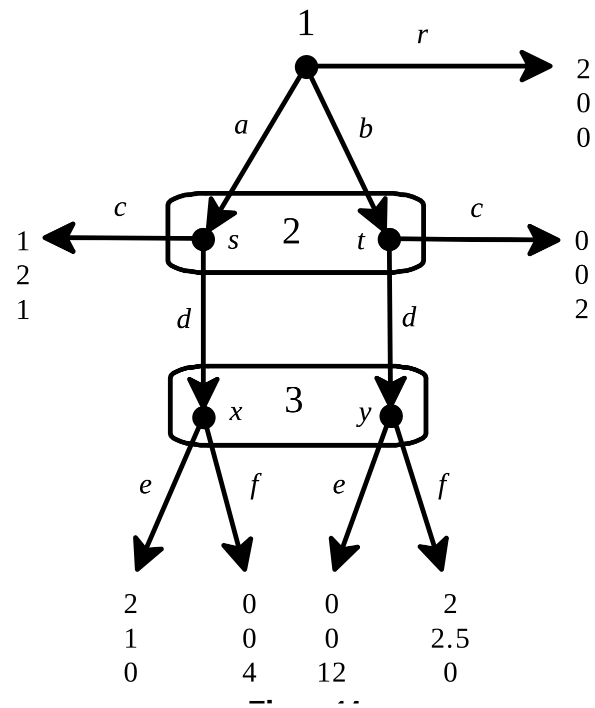

Figure 11.14: The game for Exercise 11.3.

11.4.2 Exercises for Section 11.2: Bayesian updating at reached information sets

The answers to the following exercises are in Section 11.5 at the end of this chapter.

Exercise 11.4

For the game of Figure 11.12 find a system of beliefs μ such that (σ, μ) satisfies Bayesian updating at reached information sets (Definition 11.2.2), where

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5} \end{array}\right).$$

■

Exercise 11.5

In the game shown in Figure 11.15, let

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & r & c & d & e & f \\ \frac{2}{10} & \frac{1}{10} & \frac{7}{10} & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array} \right).$$

Find all the systems of beliefs which, combined with σ yield assessments that satisfy Bayesian updating at reached information sets. ■

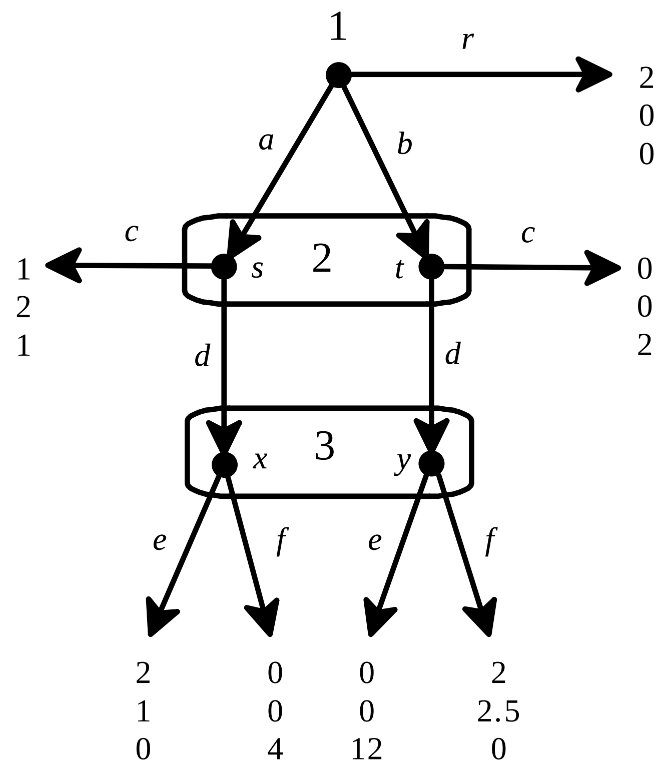

Figure 11.15: The game for Exercise 11.5.

11.4.3 Exercises for Section 11.3: Weak sequential equilibrium

The answers to the following exercises are in Section 11.5 at the end of this chapter.

Exercise 11.6

Find all the pure-strategy weak sequential equilibria of the game shown in Figure 11.16. ■

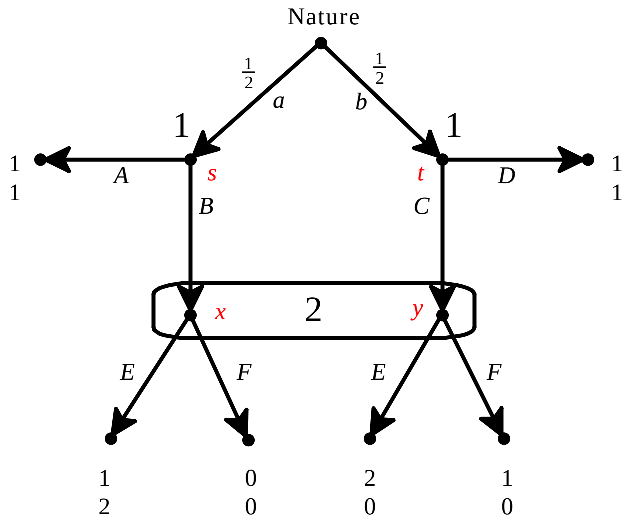

Figure 11.16: The game for Exercise 11.6.

Exercise 11.7

In the game of Figure 11.17 (which reproduces Figure 11.8), is there a pure-strategy weak sequential equilibrium where Player 1 chooses "pass"? ■

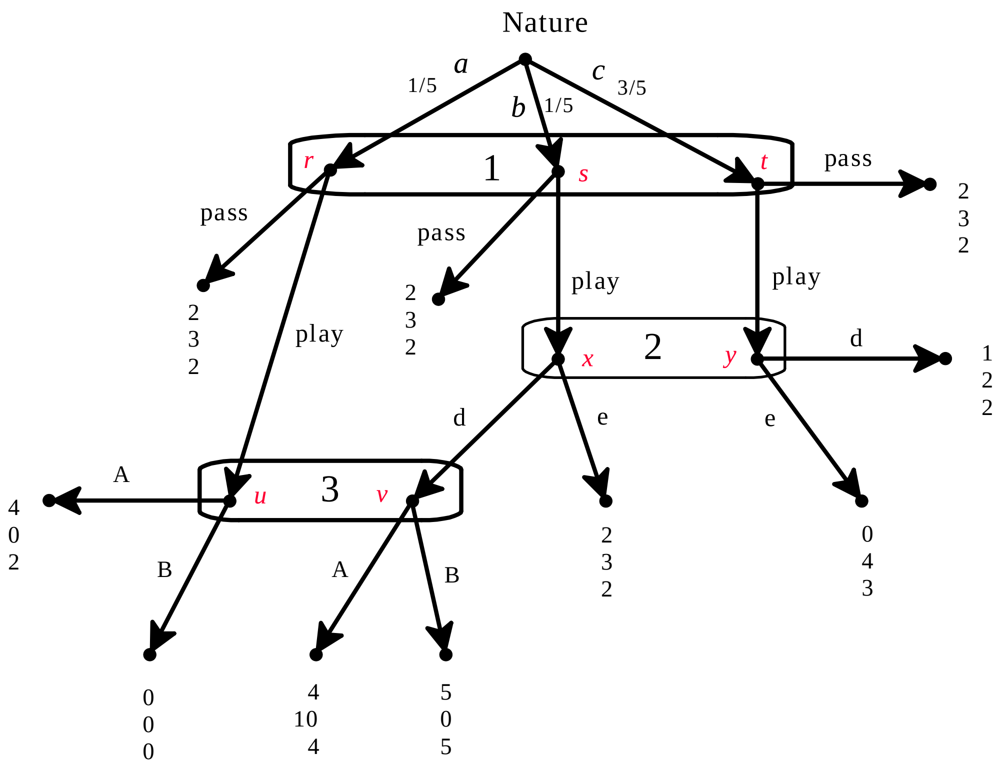

Figure 11.17: The game for Exercise 11.7.

Exercise 11.8 — ⋆⋆⋆Challenging Question ⋆⋆⋆.

Player 1 can take action C or L and Player 2 can take action c or f. The von Neumann-Morgenstern payoffs are as shown in Figure 11.18.

The game, however, is more complex than the strategic form shown in Figure 11.18.

- Player 1 moves first and chooses between C and L.
- He then sends an e-mail to Player 2 telling her truthfully what choice he made.
- However, it is commonly known between the two players that a hacker likes to intercept e-mails and change the text. The hacker is a computer program that, with probability $(1-\varepsilon)$, leaves the text unchanged and, with probability ε, changes the sentence "I chose C" into the sentence "I chose L" and the sentence "I chose L" into the sentence "I chose C". This is commonly known.
- The value of ε is also commonly known. Assume that $\varepsilon \in \left(0, \frac{1}{4}\right)$.

(a) Draw the extensive-form game.

(b) Find all the pure-strategy weak sequential equilibria.

(c) Are there any weak sequential equilibria (pure or mixed) in which Player 2, when he receives a message from Player 1 saying "I chose L", plays f with probability 1?

■

		Player 2: c	Player 2: f
Player 1	C	4 , 4	6 , 3
	L	3 , 1	5 , 2

Figure 11.18: The payoffs for Exercise 11.8.

11.5 Solutions to Exercises

Solutions to Exercise 11.1 The assessment

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right), \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

is sequentially rational. In fact,

- at the root, a gives Player 1 a payoff of 0 and so do b and c (given the strategies of Players 2 and 3). Thus any mixture of a, b and c is sequentially rational; in particular, $\begin{pmatrix} a & b & c \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} \end{pmatrix}$;
- at Player 2's node t, given Player 3's strategy, d gives Player 2 a payoff of 1 while e gives a payoff of 2; thus e is sequentially rational;
- given μ at information set $\{x,y\}$, f gives Player 2 a payoff of $\frac{1}{3}(3)+\frac{2}{3}(3)=3$ while g gives $\frac{1}{3}(6)+\frac{2}{3}(0)=2$; thus f is sequentially rational;
- given μ at information set $\{w,z\}$, h gives Player 3 a payoff of $\frac{1}{2}(3)+\frac{1}{2}(0)=1.5$, while k gives 1; thus h is sequentially rational. □

Solutions to Exercise 11.2 It might seem that the answer is the same as in the previous exercise, because the calculations at the various information sets remain the same; however, in this game checking sequential rationality at the root involves checking whether, given the strategy of Player 2 (namely (f,e)), Player 1 can increase his payoff by changing his entire strategy, that is by changing his choices at both the root and at information set $\{w,z\}$. Indeed, if Player 1 changes his strategy from

$$\left(\begin{array}{ccc|cc} a & b & c & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 \end{array}\right) \quad \text{to} \quad \left(\begin{array}{ccc|cc} a & b & c & h & k \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

his payoff increases from 0 to 1.

Hence $\left(\begin{array}{ccc|cc} a & b & c & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 \end{array}\right)$ is not sequentially rational at the root.

□

Solutions to Exercise 11.3 The game under consideration is shown in Figure 11.19.

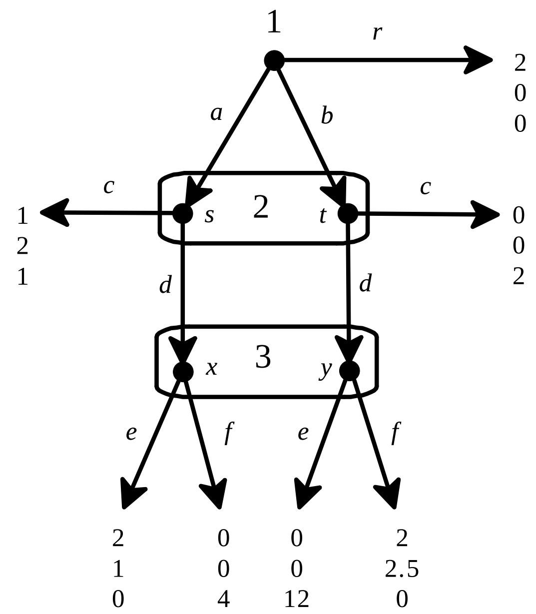

Figure 11.19: The game for Exercise 11.3.

The assessment

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & r & c & d & e & f \\ 0 & 0 & 1 & \frac{2}{5} & \frac{3}{5} & \frac{1}{3} & \frac{2}{3} \end{array} \right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} s & t & x & y \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{array} \right)$$

is sequentially rational, as the following calculations show.

- At Player 3's information set $\{x, y\}$, e gives Player 3 a payoff of $\frac{3}{4}(0) + \frac{1}{4}(12) = 3$ and f a payoff of $\frac{3}{4}(4) + \frac{1}{4}(0) = 3$;
  thus both e and f are optimal and so is any mixture of e and f;
  in particular, the mixture $\begin{pmatrix} e & f \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.
- At Player 2's information set $\{s, t\}$, c gives Player 2 a payoff of $\frac{1}{2}(2) + \frac{1}{2}(0) = 1$ and (given the strategy of Player 3)
  d a payoff of $\frac{1}{2}\left[\frac{1}{3}(1) + \frac{2}{3}(0)\right] + \frac{1}{2}\left[\frac{1}{3}(0) + \frac{2}{3}\left(\frac{5}{2}\right)\right] = 1$;
  thus both c and d are optimal and so is any mixture of c and d;
  in particular the mixture $\begin{pmatrix} c & d \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$.
- At the root, r gives Player 1 a payoff of 2
  and (given the strategies of Players 2 and 3)
  a gives a payoff of $\frac{2}{5}(1) + \frac{3}{5}\left[\frac{1}{3}(2) + \frac{2}{3}(0)\right] = \frac{4}{5}$
  and b a payoff of $\frac{2}{5}(0) + \frac{3}{5}\left[\frac{1}{3}(0) + \frac{2}{3}(2)\right] = \frac{4}{5}$. Thus r is sequentially rational. □

Solutions to Exercise 11.4 The game under consideration is shown in Figure 11.20.

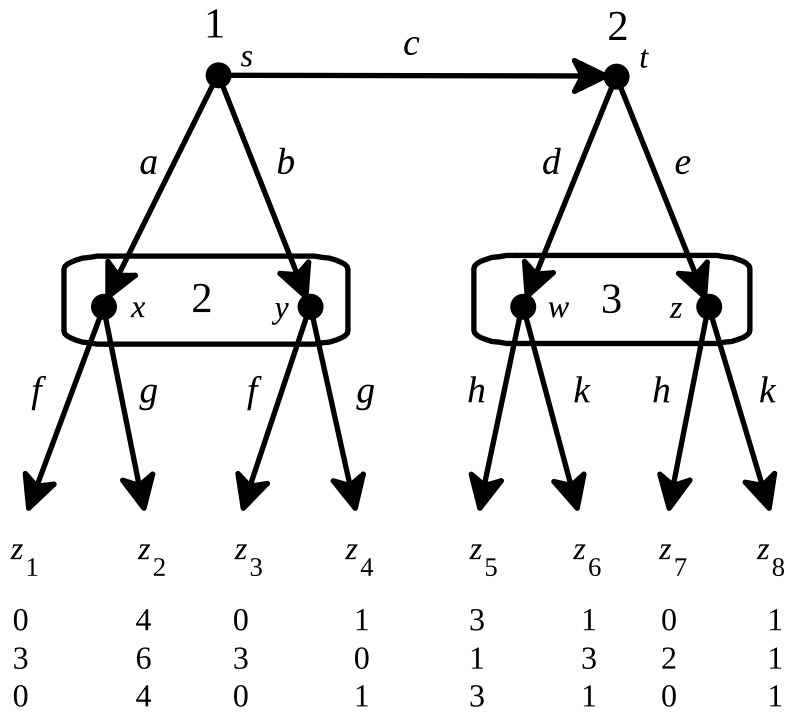

Figure 11.20: The game for Exercise 11.4.

The system of beliefs is $\mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{array}\right)$.

In fact, we have that

$\mathbb{P}_{root,\sigma}(x) = \frac{1}{8}, \quad \mathbb{P}_{root,\sigma}(y) = \frac{3}{8}, \quad \mathbb{P}_{root,\sigma}(\{x,y\}) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8},$

$\mathbb{P}_{root,\sigma}(w) = \frac{4}{8}\left(\frac{3}{4}\right) = \frac{3}{8}, \quad \mathbb{P}_{root,\sigma}(z) = \frac{4}{8}\left(\frac{1}{4}\right) = \frac{1}{8}$ and $\mathbb{P}_{root,\sigma}(\{w,z\}) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8}.$

Thus

$$\mu(x) = \frac{\mathbb{P}_{root,\sigma}(x)}{\mathbb{P}_{root,\sigma}(\{x,y\})} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}, \qquad \mu(y) = \frac{\mathbb{P}_{root,\sigma}(y)}{\mathbb{P}_{root,\sigma}(\{x,y\})} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4},$$

$$\mu(w) = \frac{\mathbb{P}_{root,\sigma}(w)}{\mathbb{P}_{root,\sigma}(\{w,z\})} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4}, \qquad \mu(z) = \frac{\mathbb{P}_{root,\sigma}(z)}{\mathbb{P}_{root,\sigma}(\{w,z\})} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$$

.

□

Solutions to Exercise 11.5 The game under consideration is shown in Figure 11.21.

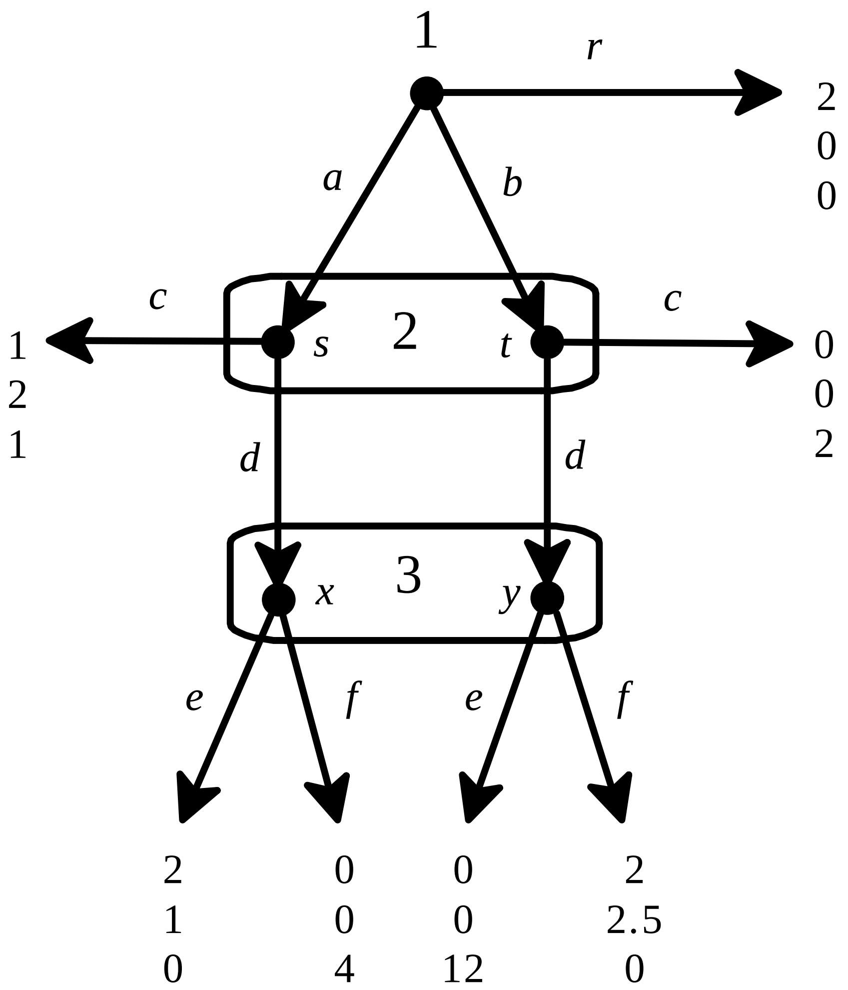

Figure 11.21: The game for Exercise 11.5.

Let

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & r & c & d & e & f \\ \frac{2}{10} & \frac{1}{10} & \frac{7}{10} & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array} \right).$$

Since only information set $\{s,t\}$ is reached by σ, no restrictions are imposed on the beliefs at information set $\{x,y\}$.
Thus, for every p such that $0 \leq p \leq 1$, the system of beliefs

$$\mu = \left(\begin{array}{cc|cc} s & t & x & y \\ \frac{2}{3} & \frac{1}{3} & p & 1-p \end{array} \right)$$

combined with σ yields an assessment that satisfies Bayesian updating at reached information sets. □

Solutions to Exercise 11.6 The game under consideration is shown in Figure 11.22.

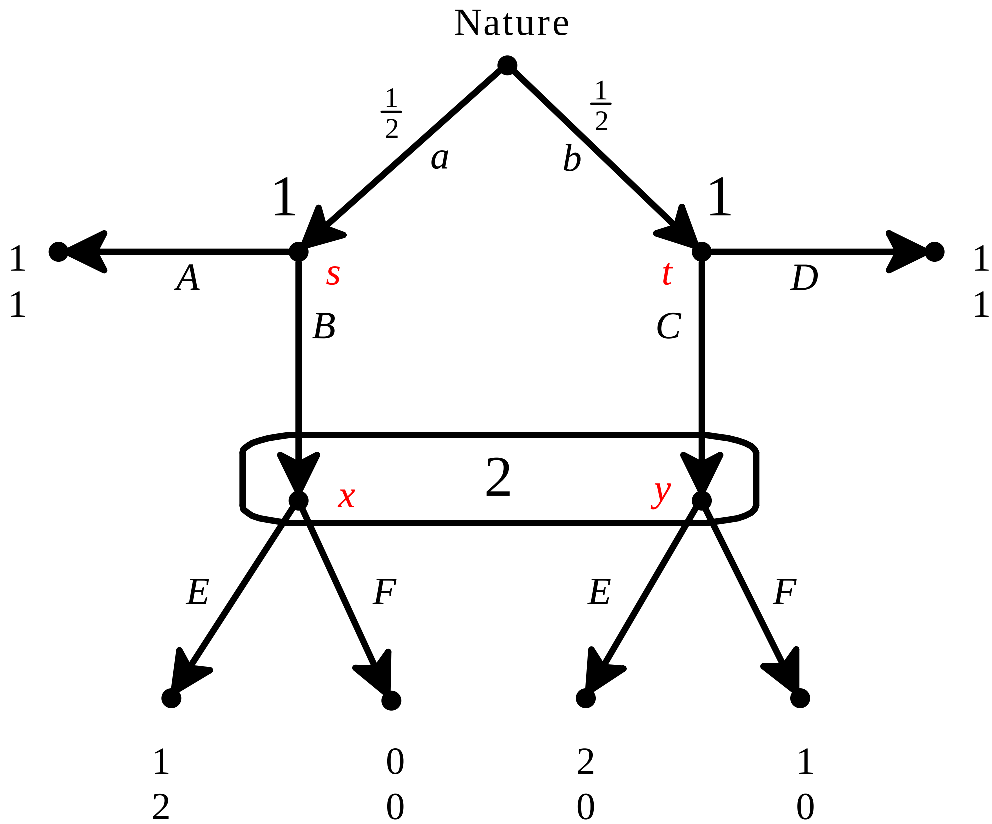

Figure 11.22: The game for Exercise 11.6.

We restrict attention to pure strategies. First of all, note that – for Player 1 – A is sequentially rational no matter what strategy Player 2 chooses and, similarly, C is sequentially rational no matter what strategy Player 2 chooses. The strategy of Player 1 determines the beliefs of Player 2 at her information set $\{x,y\}$.

Let us consider the four possibilities (recall that $S_1 = \{(A,C),(B,D),(B,C),(A,D)\}$).

- If Player 1's strategy is (A,C), then Player 2's information set $\{x,y\}$ is reached with positive probability and the only beliefs that are consistent with Bayesian updating are $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$,
  so that both E and F are sequentially rational for Player 2.
  By our preliminary observation it follows that $((A,C),E)$ with beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is a weak sequential equilibrium and so is $((A,C),F)$ with beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$.

- If Player 1's strategy is (B,D), then Player 2's information set $\{x,y\}$ is reached with positive probability and the only beliefs that are consistent with Bayesian updating are $\begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}$,
  so that – by sequential rationality – Player 2 must choose E.
  However, if Player 2's strategy is E then at node t it is not sequentially rational for Player 1 to choose D.
  Thus there is no pure-strategy weak sequential equilibrium where Player 1's strategy is (B,D).

- If Player 1's strategy is (B,C), then Player 2's information set $\{x,y\}$ is reached (with probability 1) and the only beliefs that are consistent with Bayesian updating are $\begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
  Given these beliefs, E is the only sequentially rational choice for Player 2 (her payoff from playing E is $\frac{1}{2}(2)+\frac{1}{2}(0)=1$, while her payoff from playing F is 0).
  Thus $((B,C),E)$ with beliefs $\begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is a weak sequential equilibrium.

- If Player 1's strategy is (A,D), then Player 2's information set $\{x,y\}$ is not reached and thus, according to the notion of weak sequential equilibrium, any beliefs are allowed there.
  In order for D to be sequentially rational for Player 1 it must be that Player 2's pure strategy is F.
  In order for F to be sequentially rational for Player 2, her beliefs must be $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$.
  Thus $((A,D),F)$ with beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is a weak sequential equilibrium.

Summarizing, there are four pure-strategy weak sequential equilibria:

1. $((A,C),E)$ with beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$,
2. $((A,C),F)$ with beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$,
3. $((B,C),E)$ with beliefs $\begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$,
4. $((A,D),F)$ with beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$. □

Solutions to Exercise 11.7 The game under consideration is shown in Figure 11.23.

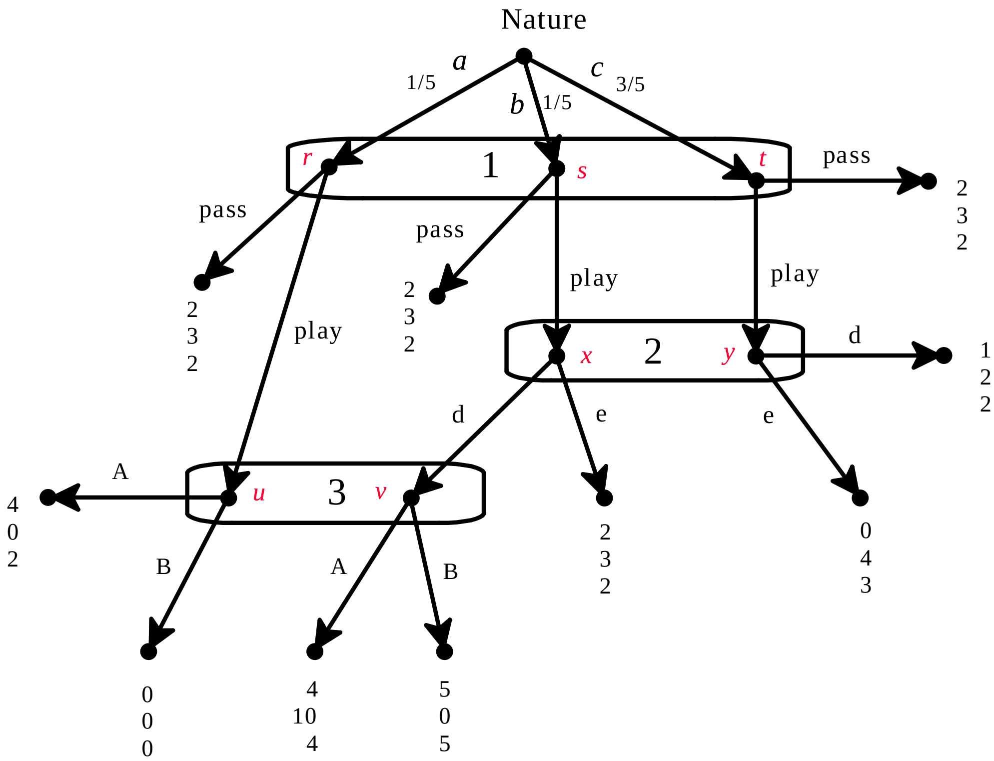

Figure 11.23: The game for Exercise 11.7.

When Player 1's strategy is "pass" then it is much easier to construct a weak sequential equilibrium, because there are no restrictions on the beliefs at the information sets of Players 2 and 3.

For example, we can choose beliefs $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ for Player 2, which make e the only sequentially rational choice, and beliefs $\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}$ for Player 3, which make B the only sequentially rational choice.

It only remains to check sequential rationality of "pass": if Player 1 chooses "pass" he gets a payoff of 2, while if he chooses "play" he gets a payoff of $\frac{1}{5}(0)+\frac{1}{5}(2)+\frac{3}{5}(0)=\frac{2}{5}<2$, so that "pass" is indeed the better choice.

Thus we have found the following weak sequential equilibrium:

$$\sigma=\left(\begin{array}{cc|cc|cc} pass & play & d & e & A & B \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \quad \text{and} \quad \mu=\left(\begin{array}{ccc|cc|cc} r & s & t & x & y & u & v \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & 1 & 0 & 1 \end{array}\right).$$

(Note, however, that this is just one of several weak sequential equilibria). □

Solutions to Exercise 11.8

(a) The extensive form is shown in Figure 11.24.

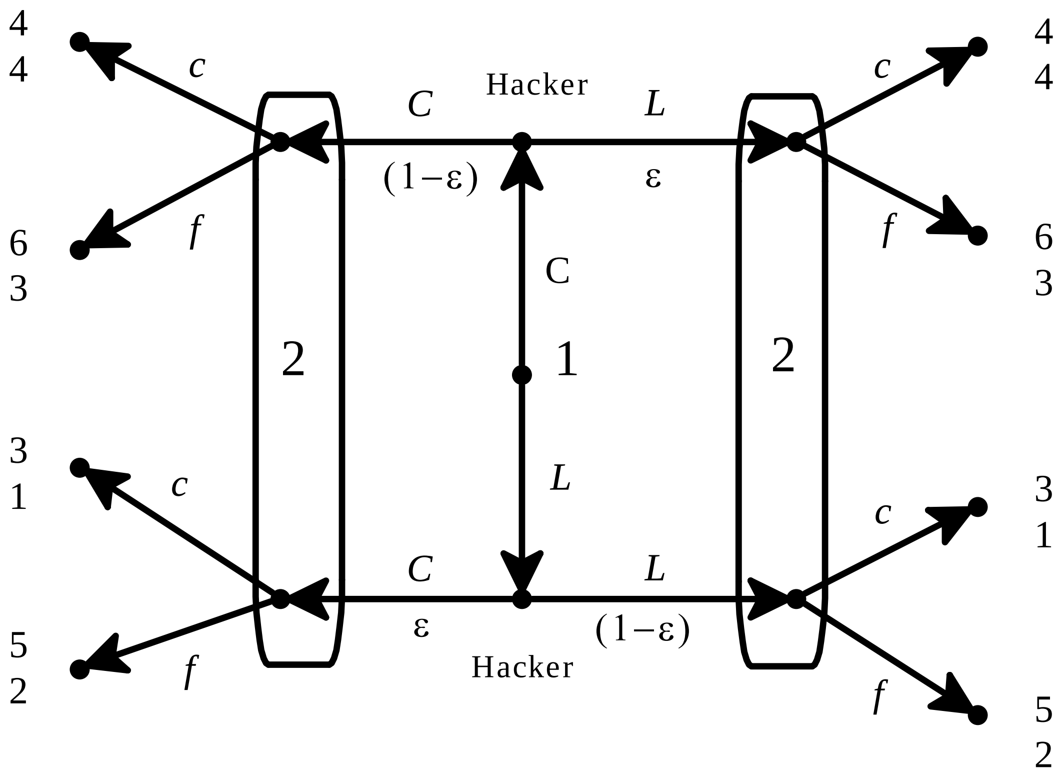

Figure 11.24: The game for Exercise 11.8.

(b) Let us consider the pure strategies of Player 1. If Player 1 plays L then Player 2 assigns probability 1 to the bottom node of each information set and responds with f with probability 1, but this makes Player 1 want to deviate to C to get a payoff of 6.[6] Thus there is no pure-strategy sequential equilibrium where Player 1 plays L.

On the other hand, if Player 1 plays C, then Player 2 assigns probability 1 to the top node of each information set and thus sequential rationality requires her to respond with c at each information set, which makes playing C optimal for Player 1. Thus $(C,(c,c))$, with beliefs that assign probability one to the top node of each information set, is the only pure-strategy weak sequential equilibrium.

[6] In other words, if Player 1 plays L, then Bayesian updating requires Player 2 to assign probability 1 to the bottom node of each information set and then sequential rationality requires Player 2 to play f at each information set, so that $(L,(f,f))$ is the only candidate for a weak sequential equilibrium where Player 1 plays L. But L is not a best reply to (f,f) and thus $(L,(f,f))$ is not a Nash equilibrium and hence cannot be part of a weak sequential equilibrium.

(c) Suppose that Player 2 plays f after reading the message "I chose L", that is, at her information set on the right. We know from the argument in part **(b)** that there are no equilibria of this kind in which Player 1 chooses a pure strategy, so Player 1 must be playing a mixed strategy. For him to be willing to do so, he must receive the same payoff from playing C or L.

If we let p be the probability with which Player 2 plays c if she receives the message "I

$$\underbrace{(1-\varepsilon)\,[4p+6(1-p)]+\varepsilon 6}_{=\pi_1(C)} \;=\; \underbrace{\varepsilon\,[3p+5(1-p)]+(1-\varepsilon)5}_{=\pi_1(L)}$$

that is, when

$$p = \frac{1}{2-4\varepsilon}.$$

Note that, since $\varepsilon \in \left(0, \frac{1}{4}\right)$, it follows that

$$p \in \left(\tfrac{1}{2}, 1\right).$$

Thus Player 2 randomizes after reading "I chose C". For Player 2 to be willing to do this, she must be indifferent between c and f in this event.

- Let $q \in (0,1)$ be the probability with which Player 1 plays C (so that $1-q$ is the probability of L);
- then Bayesian updating requires that Player 2 assign probability $\frac{(1-\varepsilon)q}{(1-\varepsilon)q+\varepsilon(1-q)}$ to the top node of her information set on the **left** and probability $\frac{\varepsilon(1-q)}{(1-\varepsilon)q+\varepsilon(1-q)}$ to the bottom node.
- Then, for Player 2, the expected payoff from playing c at the information set on the left is $\pi_2(c) = \frac{(1-\varepsilon)q}{(1-\varepsilon)q+\varepsilon(1-q)}4 + \frac{\varepsilon(1-q)}{(1-\varepsilon)q+\varepsilon(1-q)}1$ and the expected payoff from playing f is $\pi_2(f) = \frac{(1-\varepsilon)q}{(1-\varepsilon)q+\varepsilon(1-q)}3 + \frac{\varepsilon(1-q)}{(1-\varepsilon)q+\varepsilon(1-q)}2.$
- Player 2 is indifferent if these two are equal, that is if

$$4q(1-\varepsilon)+(1-q)\varepsilon = 3q(1-\varepsilon)+2(1-q)\varepsilon,$$

which is true if and only if $q = \varepsilon$.

We have now specified behavior at all information sets. To ensure that the specified behavior constitutes an equilibrium, we need to check that f is optimal for Player 2 if she receives the message "I chose L".

This will be true if

$\pi_2(c|\text{I chose L}) \leq \pi_2(f|\text{I chose L})$

$\underbrace{\text{if and only if}}_{\text{since } \varepsilon=q}$ $\quad 4q\varepsilon + 1(1-q)(1-\varepsilon) \leq 3q\varepsilon + 2(1-q)(1-\varepsilon)$

if and only if $\quad 4\varepsilon^2 + (1-\varepsilon)^2 \leq 3\varepsilon^2 + 2(1-\varepsilon)^2$

if and only if $\quad \varepsilon \leq \frac{1}{2}.$

Since we have assumed that $\varepsilon < \frac{1}{4}$, Player 2 strictly prefers to play f after receiving the message "I chose L". Thus we have constructed the following weak sequential equilibrium:

- Behavior strategy of Player 1:

$$\begin{pmatrix} C & L \\ \varepsilon & 1-\varepsilon \end{pmatrix}.$$

- Behavior strategy of Player 2: at the information set on the left (where she receives the message "I chose C"):

$$\begin{pmatrix} c & f \\ \frac{1}{2-4\varepsilon} & \frac{1-4\varepsilon}{2-4\varepsilon} \end{pmatrix}.$$

  at the information set on the right (where she receives the message "I chose L"):

$$\begin{pmatrix} c & f \\ 0 & 1 \end{pmatrix}.$$

- Player 2's beliefs *at the information set on the left* assign probability

$$\frac{\varepsilon(1-\varepsilon)}{\varepsilon(1-\varepsilon)+\varepsilon(1-\varepsilon)} = \frac{1}{2} \qquad \text{to the top node}$$

  and probability $\frac{1}{2}$ to the bottom node

  and her beliefs *at the information set on the right* assign probability

$$\frac{\varepsilon^2}{\varepsilon^2+(1-\varepsilon)^2} \qquad \text{to the top node}$$

  and probability

$$\frac{(1-\varepsilon)^2}{\varepsilon^2+(1-\varepsilon)^2} \qquad \text{to the bottom node.}$$

□