12. Sequential Equilibrium

12.1 Consistent assessments

The stated goal at the beginning of Chapter 11 was to seek a refinement of the notion of subgame-perfect equilibrium that would rule out strictly dominated choices at unreached information sets.

The notion of weak sequential equilibrium achieved the goal of ruling out strictly dominated choices, by means of the requirement of sequential rationality. According to this requirement, a choice at an information set of Player i must be optimal given Player i's beliefs at that information set; for a strictly dominated choice there can be no beliefs that make it optimal.

However, the notion of weak sequential equilibrium turned out *not* to be a refinement of subgame-perfect equilibrium: as shown in Section 11.3 (Chapter 11), it is possible for the strategy profile in a weak sequential equilibrium not to be a subgame-perfect equilibrium. The reason for this is that the only restriction on beliefs that is incorporated in the notion of weak sequential equilibrium is Bayesian updating at reached information sets. At an information set that is *not* reached by the strategy profile under consideration any beliefs whatsoever are allowed, even if those beliefs are at odds with the strategy profile.

To see this, consider the game of Figure 12.1 and the assessment consisting of the pure-strategy profile $\sigma = (c,d,f)$ (highlighted as double edges) and the system of beliefs that assigns probability 1 to node u: $\begin{pmatrix} s & t & u \\ 0 & 0 & 1 \end{pmatrix}$.

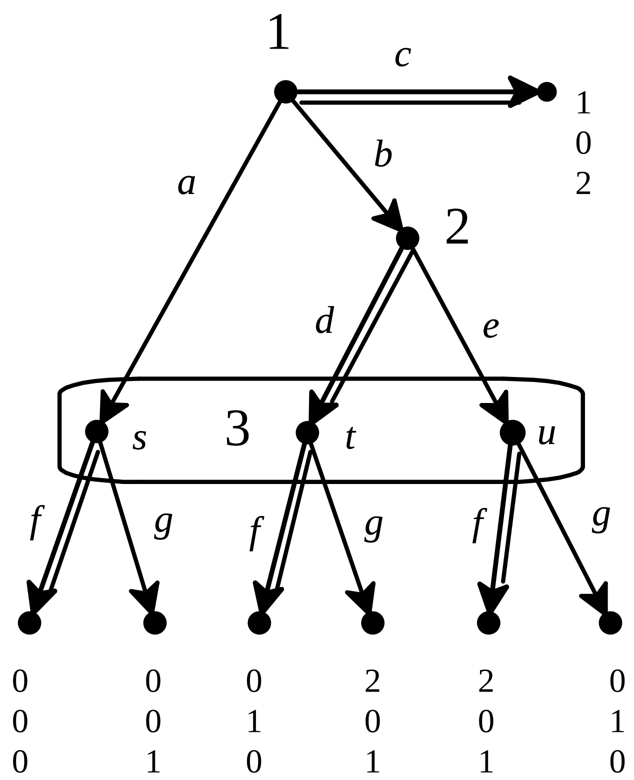

Figure 12.1: A system of beliefs μ that attaches probability 1 to node u is at odds with the strategy profile $\sigma = (c,d,f)$.

Since Player 3's information set $\{s,t,u\}$ is not reached by σ, Bayesian updating imposes no restrictions on beliefs at that information set. However, attaching probability 1 to node u is at odds with σ because in order for node u to be reached the play must have gone through Player 2's node and there, according to σ, Player 2 should have played d with probability 1, making it impossible for node u to be reached.

Thus we need to impose some restrictions on beliefs to ensure that they are consistent with the strategy profile with which they are paired (in the assessment under consideration). At reached information sets this is achieved by requiring Bayesian updating, but so far we have imposed no restriction on beliefs at unreached information sets. We want these restrictions to be "just like Bayesian updating". Kreps and Wilson (1982) proposed a restriction on beliefs that they called *consistency*, which is stated formally in Definition 12.1.1. To understand the rationale behind this notion, note that if σ is a completely mixed strategy profile (in the sense that $\sigma(a) > 0$, for every choice a) then the issue disappears, because every information set is reached with positive probability and Bayesian updating yields unique beliefs at every information set.

For example, in the game of Figure 12.1, if Player 1 uses the completely mixed strategy $\begin{pmatrix} a & b & c \\ p_a & p_b & 1-p_a-p_b \end{pmatrix}$ with $p_a, p_b \in (0,1)$ and $p_a + p_b < 1$

and Player 2 uses the completely mixed strategy $\begin{pmatrix} d & e \\ p_d & 1-p_d \end{pmatrix}$ with $p_d \in (0,1)$

then, by Bayesian updating, Player 3's beliefs must be

$$\mu(s) = \frac{p_a}{p_a + p_b p_d + p_b(1-p_d)}$$

$$\mu(t) = \frac{p_b p_d}{p_a + p_b p_d + p_b(1-p_d)}$$

$$\mu(u) = \frac{p_b(1-p_d)}{p_a + p_b p_d + p_b(1-p_d)}.$$

In the case of a completely mixed strategy profile σ, it is clear what it means for a system of beliefs μ to be consistent with the strategy profile σ: μ must be the unique system of beliefs obtained from σ by applying Bayesian updating.

What about assessments (σ, μ) where σ is such that some information sets are not reached? How can we decide, in such cases, whether μ is consistent with σ? Kreps and Wilson (1982) proposed the following criterion: the assessment (σ, μ) is consistent if there is a completely mixed strategy profile σ' which is arbitrarily close to σ and whose associated unique system of beliefs μ' (obtained by applying Bayesian updating) is arbitrarily close to μ. In mathematics "arbitrary closeness" is captured by the notion of limit.

Definition 12.1.1 Given an extensive game, an assessment (σ, μ) is *consistent* if there is a sequence of completely mixed strategy profiles $\langle \sigma_1, \sigma_2, \ldots, \sigma_n, \ldots \rangle$ such that:

1. the sequence converges to σ as n tends to infinity, that is, $\lim_{n \to \infty} \sigma_n = \sigma$, and
2. letting μ_n be the unique system of beliefs obtained from σ_n by using Bayesian updating, the sequence $\langle \mu_1, \mu_2, \ldots, \mu_n, \ldots \rangle$ converges to μ as n tends to infinity, that is, $\lim_{n \to \infty} \mu_n = \mu$.

For example, consider the extensive form shown in Figure 12.2.

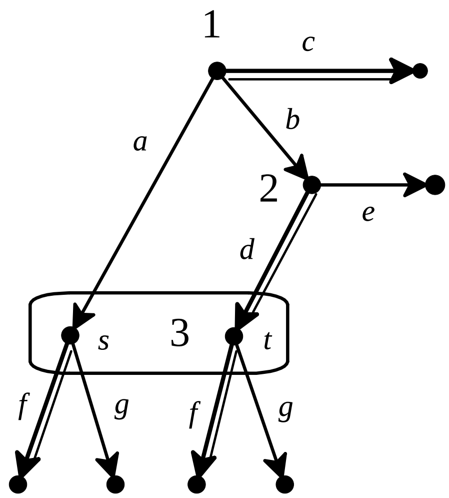

Figure 12.2: The assessment $\sigma = (c,d,f)$ and $\mu(s) = \frac{3}{8}$, $\mu(t) = \frac{5}{8}$ is consistent.

The assessment $\sigma = (c,d,f)$, $\mu = \begin{pmatrix} s & t \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$ is consistent. To see this, let

$$\sigma_n = \left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g \\ \frac{3}{n} & \frac{5}{n} & 1-\frac{8}{n} & 1-\frac{1}{n} & \frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} \end{array}\right).$$

Then, as n tends to infinity, all of $\frac{3}{n}, \frac{5}{n}, \frac{1}{n}$ tend to 0 and both $1-\frac{8}{n}$ and $1-\frac{1}{n}$ tend to 1:

$$\lim_{n\to\infty} \sigma_n = \left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right) = \sigma.$$

Furthermore, $\mu_n(s) = \dfrac{\frac{3}{n}}{\frac{3}{n}+\frac{5}{n}\left(1-\frac{1}{n}\right)} = \dfrac{3}{8-\frac{5}{n}}$, which tends to $\frac{3}{8}$ as n tends to infinity and $\mu_n(t) = \dfrac{\frac{5}{n}\left(1-\frac{1}{n}\right)}{\frac{3}{n}+\frac{5}{n}\left(1-\frac{1}{n}\right)} = \dfrac{5-\frac{5}{n}}{8-\frac{5}{n}}$, which tends to $\frac{5}{8}$ as n tends to infinity, so that $\lim_{n\to\infty} \mu_n = \begin{pmatrix} s & t \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix} = \mu$.

The notion of consistent assessment (σ,μ) was meant to capture an extension of the requirement of Bayesian updating that would apply also to information sets that have zero probability of being reached (when the play of the game is according to the strategy profile σ). However, Definition 12.1.1 is rather technical and not easy to apply. Showing that an assessment *is* consistent requires displaying an appropriate sequence and showing that the sequence converges to the given assessment. This is relatively easy as compared to the considerably more difficult task of proving that an assessment is *not* consistent: this requires showing that *every possible sequence* that one could construct would not converge.

To see the kind of reasoning involved, consider the extensive form of Figure 12.3.

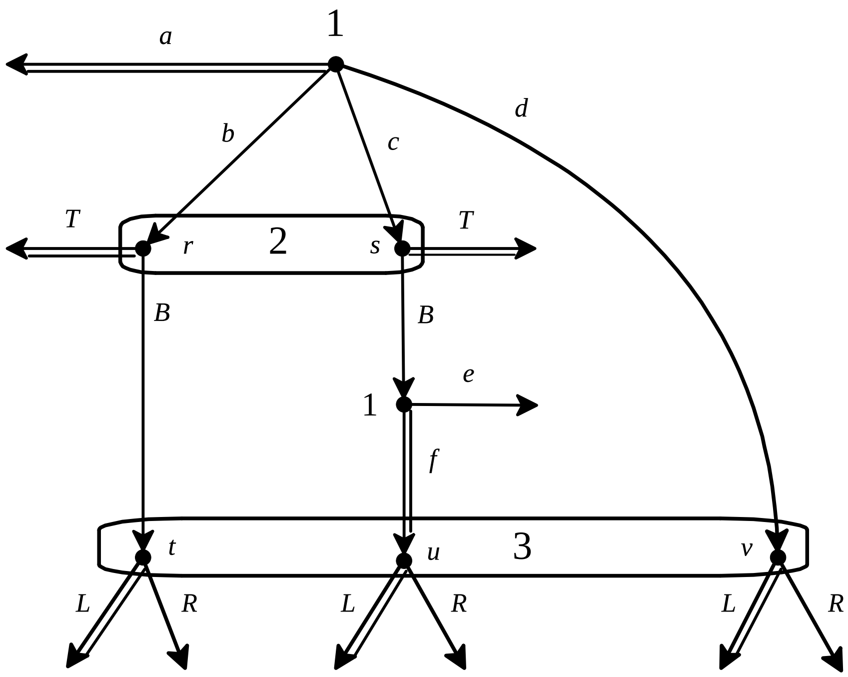

Figure 12.3: The assessment $\sigma = (a, T, f, L)$, $\mu(r) = \mu(s) = \frac{1}{2}$, $\mu(t) = \frac{1}{5}$, $\mu(u) = \mu(v) = \frac{2}{5}$ is not consistent.

We want to show that the following assessment is *not* consistent:

$$\sigma = \left(\begin{array}{cccc|cc|cc|cc} a & b & c & d & T & B & e & f & L & R \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{array} \right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|ccc} r & s & t & u & v \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{array} \right)$$

Suppose that (σ, μ) is consistent. Then there must be a sequence $\langle \sigma_n \rangle_{n=1,2,\ldots}$ of completely mixed strategies that converges to σ, whose corresponding sequence of systems of beliefs $\langle \mu_n \rangle_{n=1,2,\ldots}$ (obtained by applying Bayesian updating) converges to μ.
Let the n^{th} term of this sequence of completely mixed strategies be:

$$\sigma_n = \left(\begin{array}{cccc|cc|cc|cc} a & b & c & d & T & B & e & f & L & R \\ p_n^a & p_n^b & p_n^c & p_n^d & p_n^T & p_n^B & p_n^e & p_n^f & p_n^L & p_n^R \end{array} \right).$$

Thus $p_n^a + p_n^b + p_n^c + p_n^d = p_n^T + p_n^B = p_n^e + p_n^f = p_n^L + p_n^R = 1$,

$p_n^x \in (0,1)$ for all $x \in \{a, b, c, d, B, T, e, f, L, R\}$,

$\lim_{n\to\infty} p_n^x = 0$ for $x \in \{b, c, d, B, e, R\}$

and $\lim_{n\to\infty} p_n^x = 1$ for $x \in \{a, T, f, L\}$.

From $\sigma_n = \left(\begin{array}{cccc|cc|cc|cc} a & b & c & d & T & B & e & f & L & R \\ p_n^a & p_n^b & p_n^c & p_n^d & p_n^T & p_n^B & p_n^e & p_n^f & p_n^L & p_n^R \end{array} \right)$ we obtain, by Bayesian updating, the following system of beliefs μ_n:

$$\begin{pmatrix} r & s \\ \frac{p_n^b}{p_n^b+p_n^c} & \frac{p_n^c}{p_n^b+p_n^c} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t & u & v \\ \frac{p_n^b p_n^B}{p_n^b p_n^B + p_n^c p_n^B p_n^f + p_n^d} & \frac{p_n^c p_n^B p_n^f}{p_n^b p_n^B + p_n^c p_n^B p_n^f + p_n^d} & \frac{p_n^d}{p_n^b p_n^B + p_n^c p_n^B p_n^f + p_n^d} \end{pmatrix}$$

Note that $\frac{\mu_n(s)}{\mu_n(r)} = \frac{p_n^c}{p_n^b}$.

By hypothesis, $\lim_{n\to\infty} \mu_n(s) = \mu(s) = \frac{1}{2}$ and $\lim_{n\to\infty} \mu_n(r) = \mu(r) = \frac{1}{2}$ and thus

$$\lim_{n\to\infty} \frac{p_n^c}{p_n^b} = \lim_{n\to\infty} \frac{\mu_n(s)}{\mu_n(r)} = \frac{\lim_{n\to\infty} \mu_n(s)}{\lim_{n\to\infty} \mu_n(r)} = \frac{\mu(s)}{\mu(r)} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

On the other hand, $\frac{\mu_n(u)}{\mu_n(t)} = \frac{p_n^c}{p_n^b} p_n^f$, so that $\lim_{n\to\infty} \frac{\mu_n(u)}{\mu_n(t)} = \lim_{n\to\infty} \left(\frac{p_n^c}{p_n^b} p_n^f \right)$.

By hypothesis, $\lim_{n\to\infty} \mu_n(u) = \mu(u) = \frac{2}{5}$ and $\lim_{n\to\infty} \mu_n(t) = \mu(t) = \frac{1}{5}$, so that

$$\lim_{n\to\infty} \frac{\mu_n(u)}{\mu_n(t)} = \frac{\lim_{n\to\infty} \mu_n(u)}{\lim_{n\to\infty} \mu_n(t)} = \frac{\mu(u)}{\mu(t)} = 2.$$

However, $\lim_{n\to\infty} \left(\frac{p_n^c}{p_n^b} p_n^f \right) = \underbrace{\left(\lim_{n\to\infty} \frac{p_n^c}{p_n^b} \right)}_{=1} \underbrace{\left(\lim_{n\to\infty} p_n^f \right)}_{=0} = (1)(0) = 0$, yielding a contradiction.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 12.4.1 at the end of this chapter.

12.2 Sequential equilibrium

The notion of sequential equilibrium was introduced by Kreps and Wilson (1982).

Definition 12.2.1 Given an extensive game, an assessment (σ, μ) is a *sequential equilibrium* if it is consistent (Definition 12.1.1) and sequentially rational (Definition 11.1.2, Chapter 11).

For an example of a sequential equilibrium consider the extensive game of Figure 12.4.

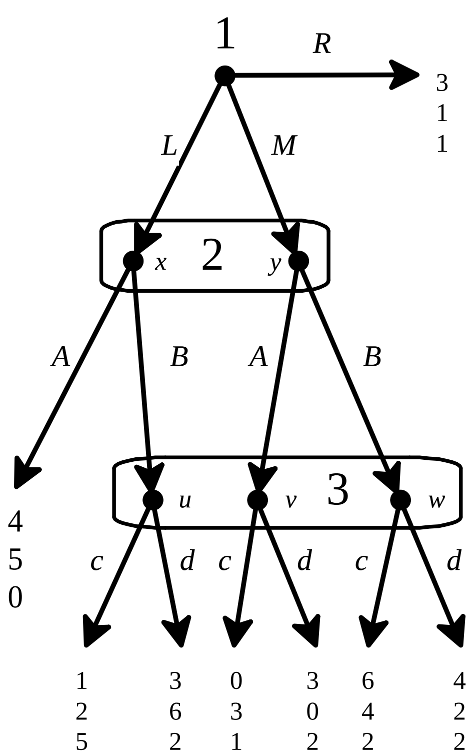

Figure 12.4: An extensive-form game with cardinal payoffs.

Let us show that the following assessment is a sequential equilibrium:

$$\sigma = \left(\begin{array}{ccc|cc|cc} L & M & R & A & B & c & d \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{array} \right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|ccc} x & y & u & v & w \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \end{array} \right).$$

Let us first verify sequential rationality of Player 3's strategy. At her information set – given her beliefs – c gives a payoff of $\frac{1}{7}(5) + \frac{3}{7}(1) + \frac{3}{7}(2) = 2$ and d gives a payoff of $\frac{1}{7}(2) + \frac{3}{7}(2) + \frac{3}{7}(2) = 2$. Thus c is optimal (as would be d and any randomization over c and d).

For Player 2, at his information set $\{x,y\}$ – given his beliefs and given the strategy of Player 3 – A gives a payoff of $\frac{1}{4}(5) + \frac{3}{4}(3) = 3.5$ and B gives a payoff of $\frac{1}{4}(2) + \frac{3}{4}(4) = 3.5$; thus any mixture of A and B is optimal, in particular, the mixture $\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is optimal.

Finally, at the root, R gives Player 1 a payoff of 3, L a payoff of $\frac{1}{2}(4) + \frac{1}{2}(1) = 2.5$ and M a payoff of $\frac{1}{2}(0) + \frac{1}{2}(6) = 3$; thus R is optimal (as would be any mixture of M and R).

Next we show consistency. Consider the sequence of completely mixed strategies $\langle\sigma_n\rangle_{n=1,2,\ldots}$ where

$$\sigma_n = \left(\begin{array}{ccc|cc|cc} L & M & R & A & B & c & d \\ \frac{1}{n} & \frac{3}{n} & 1-\frac{4}{n} & \frac{1}{2} & \frac{1}{2} & 1-\frac{1}{n} & \frac{1}{n} \end{array}\right).$$

Clearly $\lim\limits_{n\to\infty} \sigma_n = \sigma$. The corresponding sequence of systems of beliefs $\langle\mu_n\rangle_{n=1,2,\ldots}$ is given by the following constant sequence, which obviously converges to μ:

$$\mu_n = \left(\begin{array}{cc|ccc} x & y & u & v & w \\ \frac{\frac{1}{n}}{\frac{1}{n}+\frac{3}{n}} = \frac{1}{4} & \frac{\frac{3}{n}}{\frac{1}{n}+\frac{3}{n}} = \frac{3}{4} & \frac{\frac{1}{n}\left(\frac{1}{2}\right)}{\frac{1}{n}\left(\frac{1}{2}\right)+\frac{3}{n}\left(\frac{1}{2}\right)+\frac{3}{n}\left(\frac{1}{2}\right)} = \frac{1}{7} & \frac{\frac{3}{n}\left(\frac{1}{2}\right)}{\frac{1}{n}\left(\frac{1}{2}\right)+\frac{3}{n}\left(\frac{1}{2}\right)+\frac{3}{n}\left(\frac{1}{2}\right)} = \frac{3}{7} & \frac{3}{7} \end{array}\right).$$

R Since consistency implies Bayesian updating at reached information sets, every sequential equilibrium is a weak sequential equilibrium.

We now turn to the properties of sequential equilibria.

Theorem 12.2.1 — Kreps and Wilson, 1982. Given an extensive game, if (σ, μ) is a sequential equilibrium then σ is a subgame-perfect equilibrium.

Theorem 12.2.2 — Kreps and Wilson, 1982. Every finite extensive-form game with cardinal payoffs has at least one sequential equilibrium.

By Theorem 12.1, the notion of sequential equilibrium achieves the objective of refining the notion of subgame-perfect equilibrium. The relationship between the various solution concepts considered so far is shown in the Venn diagram of Figure 12.5.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 12.4.2 at the end of this chapter.

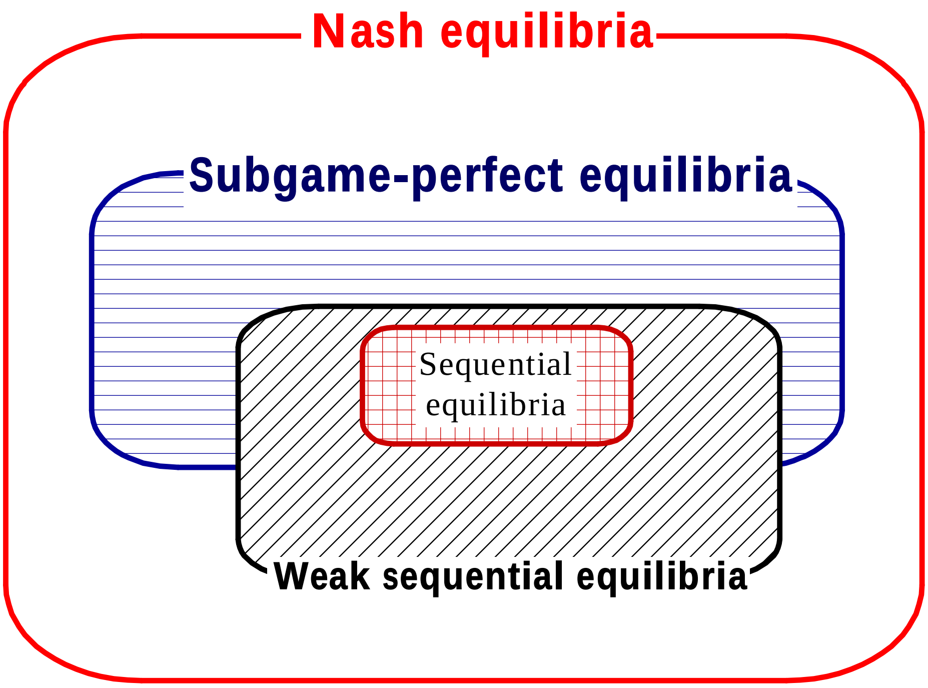

Figure 12.5: The relationship between Nash equilibrium, subgame-perfect equilibrium, weak sequential equilibrium and sequential equilibrium.

12.3 Is 'consistency' a good notion?

The notion of consistency (Definition 12.1) is unsatisfactory in two respects.

- ◇ From a practical point of view, consistency is computationally hard to prove, since one has to construct a sequence of completely mixed strategies, calculate the corresponding Bayesian beliefs and take the limit of the two sequences. The larger and more complex the game, the harder it is to establish consistency.

- ◇ From a conceptual point of view, it is not clear how one should interpret, or justify, the requirement of taking the limit of sequences of strategies and beliefs.

Concerning the latter point, Kreps and Wilson themselves express dissatisfaction with their definition of sequential equilibrium:

> "We shall proceed here to develop the properties of sequential equilibrium as defined above; however, we do so with some doubts of our own concerning what 'ought' to be the definition of a consistent assessment that, with sequential rationality, will give the 'proper' definition of a sequential equilibrium." (Kreps and Wilson, 1982, p. 876.)

In a similar vein, Osborne and Rubinstein (1994, p. 225) write

> "We do not find the consistency requirement to be natural, since it is stated in terms of limits; it appears to be a rather opaque technical assumption."

In the next chapter we will introduce a simpler refinement of subgame-perfect equilibrium which has a clear interpretation in terms of the AGM theory of belief revision: we call it *Perfect Bayesian Equilibrium*. We will also show that one can use this notion to provide a characterization of sequential equilibrium that does not require the use of limits of sequences of completely mixed strategies.

We conclude this chapter by observing that while the notion of sequential equilibrium eliminates strictly dominated choices at information sets (even if they are reached with zero probability), it is not strong enough to eliminate *weakly* dominated choices. To see this, consider the "simultaneous" game shown in Figure 12.6.

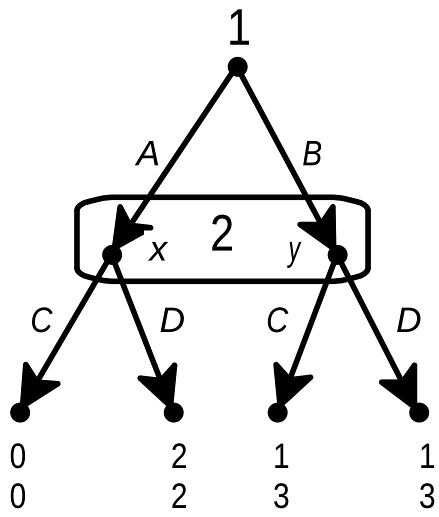

Figure 12.6: A game where a weakly dominated strategy can be part of a sequential equilibrium.

In this game there are two Nash (and subgame-perfect) equilibria: (A,D) and (B,C). Note that C is a weakly dominated strategy for Player 2. The only beliefs of Player 2 that rationalize choosing C is that Player 1 chose B with probability 1 (if Player 2 attaches any positive probability, no matter how small, to Player 1 choosing A, then D is the only sequentially rational choice). Nevertheless, both Nash equilibria are sequential equilibria. For example, it is straightforward to check that the "unreasonable" Nash equilibrium $\sigma = (B,C)$, when paired with beliefs $\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, constitutes a sequential equilibrium. Consistency of this assessment is easily verified by considering the sequence $\sigma_n = \left(\begin{array}{cc|cc} A & B & C & D \\ \frac{1}{n} & 1-\frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} \end{array} \right)$ whose associated beliefs are $\mu_n = \begin{pmatrix} x & y \\ \frac{1}{n} & 1-\frac{1}{n} \end{pmatrix}$ and sequential rationality is clearly satisfied.

Many game theorists feel that it is "irrational" to choose a weakly dominated strategy; thus further refinements beyond sequential equilibrium have been proposed. A stronger notion of equilibrium, which is a strict refinement of sequential equilibrium, is the notion of *trembling-hand perfect equilibrium.* This notion, due to Reinhardt Selten (who also introduced the notion of subgame-perfect equilibrium) precedes chronologically the notion of sequential equilibrium (Selten, 1975). *Trembling-hand perfect equilibrium* does in fact eliminate weakly dominated strategies. This topic is outside the scope of this book.[1]

12.4 Exercises

12.4.1 Exercises for Section 12.1: Consistent assessments

The answers to the following exercises are in Section **??** at the end of this chapter.

Exercise 12.1

Consider the extensive form shown in Figure 12.7.

Consider the following (partial) behavior strategy profile $\sigma = \left(\begin{array}{ccc|cc} a & b & c & d & e \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{4} & \frac{3}{4} \end{array}\right)$.

Find the corresponding system of beliefs obtained by Bayesian updating. ■

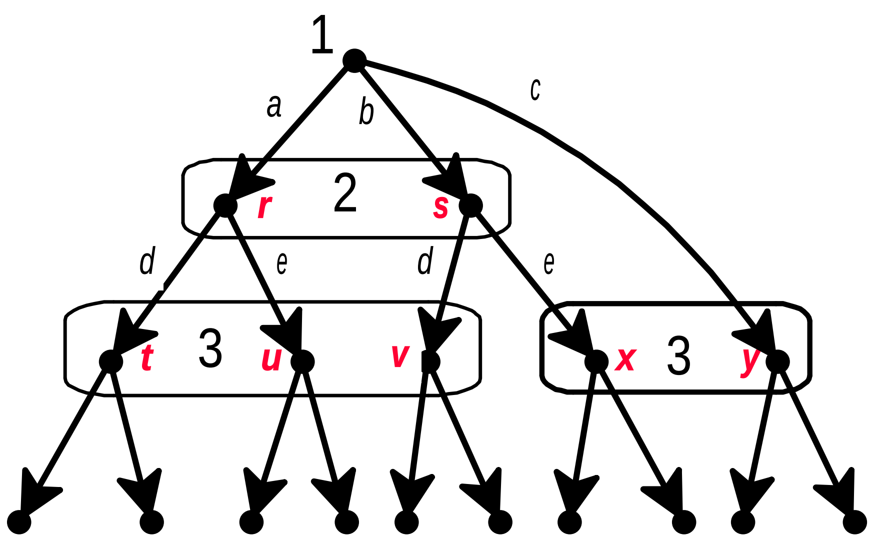

Figure 12.7: The extensive form for Exercise 12.1.

[1]The interested reader is referred to van Damme (2002).

Exercise 12.2

Consider the extensive game shown in Figure 12.8.

(a) Write the corresponding strategic form.

(b) Find all the pure-strategy Nash equilibria.

(c) Find all the pure-strategy subgame-perfect equilibria.

(d) Which of the pure-strategy subgame-perfect equilibria can be part of a consistent assessment? Give a proof for each of your claims.

■

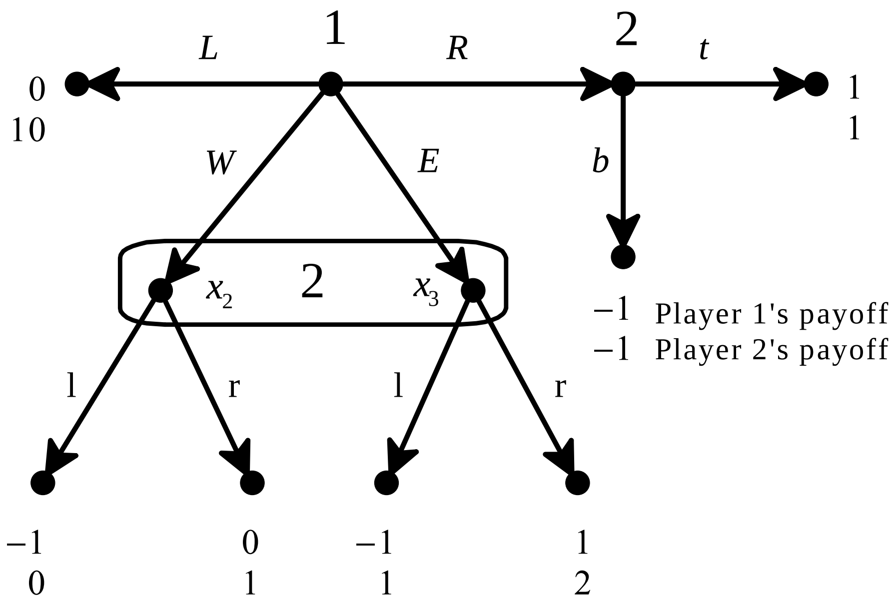

Figure 12.8: The extensive form for Exercise 12.2.

12.4.2 Exercises for Section 12.2: Sequential equilibrium

Exercise 12.3

Consider the extensive-form game shown in Figure 12.9. For each pure-strategy Nash equilibrium determine whether it is part of an assessment which is a sequential equilibrium. ■

Exercise 12.4

Consider the game shown in Figure 12.10.

(a) Find three subgame-perfect equilibria. [Use pure strategies wherever possible.]

(b) For each of the equilibria you found in Part **(a)**, explain if it can be written as part of a weak sequential equilibrium.

(c) Find a sequential equilibrium. [Use pure strategies wherever possible.]

■

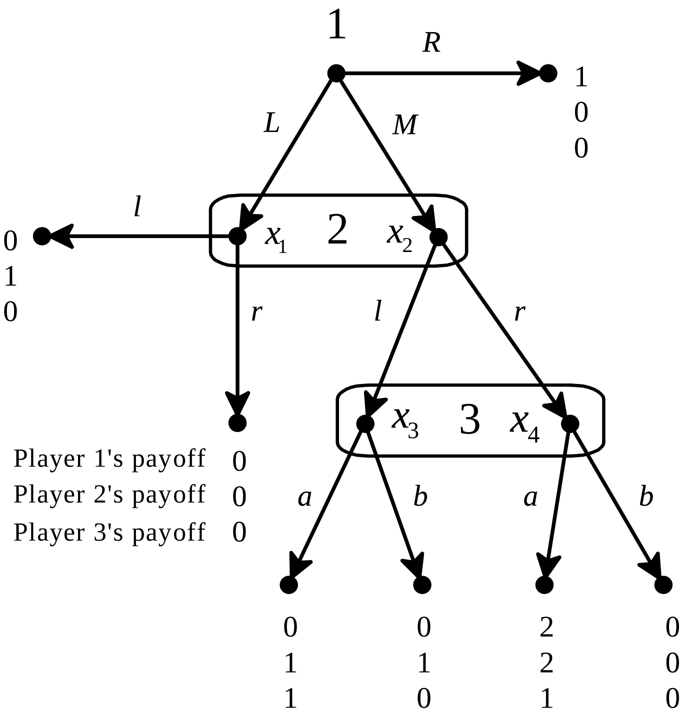

Figure 12.9: The extensive-form game for Exercise 12.3.

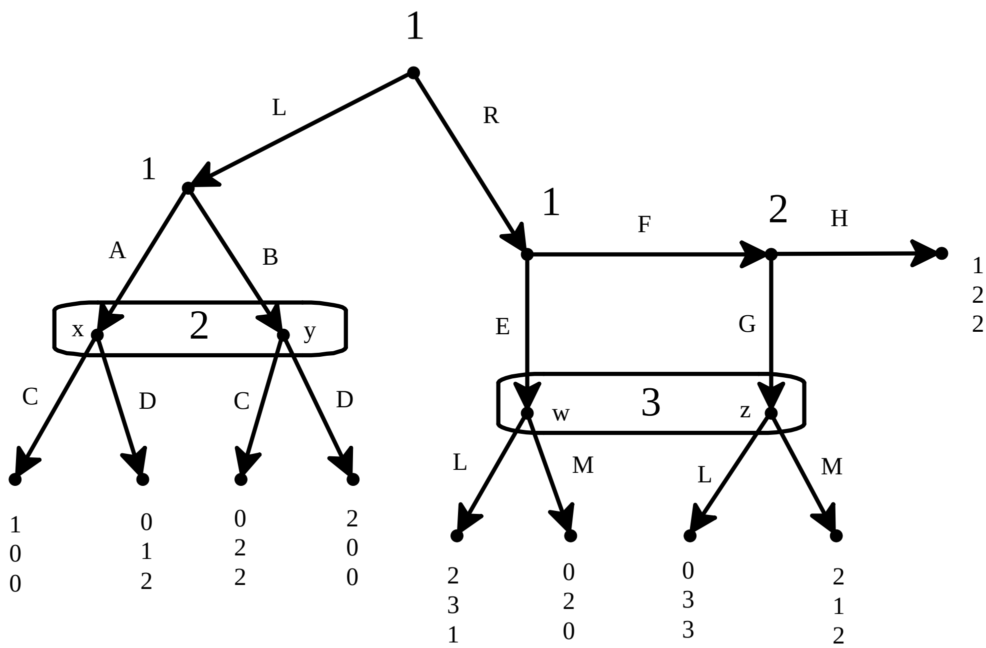

Figure 12.10: The extensive-form game for Exercise 12.4

Exercise 12.5

An electric circuit connects two switches and a light. One switch is in Room 1, the second switch is in Room 2 and the light is in Room 3.

Player 1 is in Room 1, Player 2 in Room 2 and Player 3 in Room 3.

The two switches are now in the Off position. The light in Room 3 comes on if and only if **both** switches are in the On position. Players 1 and 2 act simultaneously and independently: each is allowed only two choices, namely whether to leave her switch in the Off position or turn it to the On position.

If the light comes on in Room 3 then the game ends and Players 1 and 2 get \$100 each while Player 3 gets \$300.

If the light in Room 3 stays off, then Player 3 (not knowing what the other players did) has to make a guess as to what Players 1 and 2 did (thus, for example, one possible guess is "both players left their respective switches in the Off position").

Then the payoffs are as follows:

- if Player 3's guess turns out to be correct then each player gets \$200,
- if Player 3 makes one correct guess but the other wrong (e.g. he guesses that both Player 1 and Player 2 chose "Off" and, as a matter of fact, Player 1 chose "Off" while Player 2 chose "On"), then Player 3 gets \$50, the player whose action was guessed correctly gets \$100 and the remaining player gets nothing (in the previous example, Player 1 gets \$100, Player 2 gets nothing and Player 3 gets \$50) and
- if Player 3's guess is entirely wrong then all the players get nothing.

All the players are selfish and greedy (that is, each player only cares about how much money he/she gets and prefers more money to less) and risk neutral.

(a) Represent this situation as an extensive-form game.

(b) Write the corresponding strategic form, assigning the rows to Player 1, the columns to Player 2, etc.

(c) Find all the pure-strategy Nash equilibria.

(d) For at least one pure-strategy Nash equilibrium prove that it cannot be part of a sequential equilibrium.

■

Exercise 12.6 — ⋆⋆⋆Challenging Question ⋆⋆⋆.

A buyer and a seller are bargaining over an object owned by the seller. The value of the object to the buyer is known to her but not to the seller. The buyer is drawn randomly from a population with the following characteristics: the fraction λ value the object at $\$H$ while the remaining fraction value the object at $\$L$, with $H > L > 0$.

The bargaining takes place over two periods. In the first period the seller makes a take-it-or-leave-it offer (i.e. names the price) and the buyer accepts or rejects. If the buyer accepts, the transaction takes place and the game ends. If the buyer rejects, then the seller makes a new take-it-or-leave-it offer and the buyer accepts or rejects. In either case the game ends.

Payoffs are as follows:

(1) if the seller's offer is accepted (whether it was made in the first period or in the second period), the seller's payoff is equal to the price agreed upon and the buyer's payoff is equal to the difference between the value of the object to the buyer and the price paid;

(2) if the second offer is rejected both players get a payoff of 0.

Assume that both players discount period 2 payoffs with a discount factor $\delta \in (0,1)$, that is, from the point of view of period 1, getting $\$x$ in period 2 is considered to be the same as getting $\$\delta x$ in period 1. For example, if the seller offers price p in the first period and the offer is accepted, then the seller's payoff is p, whereas if the same price p is offered and accepted in the second period, then the seller's payoff, viewed from the standpoint of period 1, is $\$\delta p$.

Assume that these payoffs are von Neumann-Morgenstern payoffs; assume further that $H = 20$, $L = 10$, $\delta = \frac{3}{4}$ and $\lambda = \frac{2}{3}$.

(a) Draw the extensive form of this game for the case where, in both periods, the seller can only offer one of two prices: \$10 or \$12. Nature moves first and selects the value for the buyer; the buyer is informed, while the seller is not. It is common knowledge between buyer and seller that Nature will pick H with probability λ and L with probability (1-λ).

(b) For the game of part **(a)** find a pure-strategy sequential equilibrium. Prove that what you suggest is indeed a sequential equilibrium.

■

12.5 Solutions to Exercises

Solutions to Exercise 12.1 The extensive form under consideration is shown in Figure 12.11.

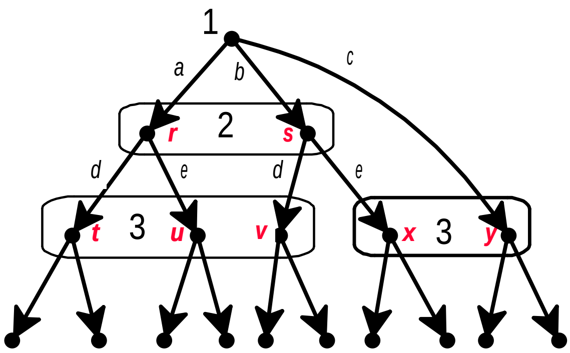

Figure 12.11: The extensive form for Exercise 12.1.

The system of beliefs obtained, by Bayesian updating, from the (partial) behavior strategy profile $\sigma = \left(\begin{array}{ccc|cc} a & b & c & d & e \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{4} & \frac{3}{4} \end{array}\right)$ is as follows:

$$\mu(r) = \frac{\frac{1}{5}}{\frac{1}{5}+\frac{3}{5}} = \frac{1}{4}, \ \ \mu(s) = \frac{\frac{3}{5}}{\frac{1}{5}+\frac{3}{5}} = \frac{3}{4}, \quad \mu(x) = \frac{\frac{3}{5}\left(\frac{3}{4}\right)}{\frac{3}{5}\left(\frac{3}{4}\right)+\frac{1}{5}} = \frac{9}{13}, \ \ \mu(y) = \frac{\frac{1}{5}}{\frac{3}{5}\left(\frac{3}{4}\right)+\frac{1}{5}} = \frac{4}{13}$$

$$\mu(t) = \frac{\frac{1}{5}\left(\frac{1}{4}\right)}{\frac{1}{5}\left(\frac{1}{4}\right)+\frac{1}{5}\left(\frac{3}{4}\right)+\frac{3}{5}\left(\frac{1}{4}\right)} = \frac{1}{7} \quad \mu(u) = \frac{\frac{1}{5}\left(\frac{3}{4}\right)}{\frac{1}{5}\left(\frac{1}{4}\right)+\frac{1}{5}\left(\frac{3}{4}\right)+\frac{3}{5}\left(\frac{1}{4}\right)} = \frac{3}{7}$$

$$\mu(v) = \frac{\frac{3}{5}\left(\frac{1}{4}\right)}{\frac{1}{5}\left(\frac{1}{4}\right)+\frac{1}{5}\left(\frac{3}{4}\right)+\frac{3}{5}\left(\frac{1}{4}\right)} = \frac{3}{7}.$$ □

Solutions to Exercise 12.2

(a) The game under consideration is shown in Figure 12.12. The corresponding strategic form is shown in Figure 12.13.

(b) The pure-strategy Nash equilibria are: (L,bl), (R,tl), (R,tr), (E,tr), (E,br).

(c) There is only one proper subgame, namely the one that starts at node x_1 ; in that subgame the unique Nash equilibrium is t.
Thus only the following are subgame-perfect equilibria: (R,tl), (R,tr), (E,tr).

(d) Each of the above subgame-perfect equilibria can be part of a consistent assessment. Let $\sigma = (R,tl)$, $\mu = \left(\begin{array}{cc} x_2 & x_3 \\ \frac{1}{2} & \frac{1}{2} \end{array}\right)$; this assessment is consistent as the following sequences of completely mixed strategies and corresponding system of beliefs show:

$$\sigma_n = \left(\begin{array}{cccc|cc|cc} L & W & E & R & b & t & l & r \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & 1-\frac{3}{n} & \frac{1}{n} & 1-\frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} \end{array}\right),$$

$$\mu_n = \left(\begin{array}{cc} x_2 & x_3 \\ \frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{1}{2} & \frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{1}{2} \end{array}\right).$$ Clearly $\lim_{n\to\infty}\sigma_n = \sigma$ and $\lim_{n\to\infty}\mu_n = \mu$.

The proof for (R,tr) is similar.

Let $\sigma = (E,tr)$, $\mu = \begin{pmatrix} x_2 & x_3 \\ 0 & 1 \end{pmatrix}$; this assessment is consistent as the following sequences of completely mixed strategies and corresponding beliefs show:

$$\sigma_n = \left(\begin{array}{cccc|cc|cc} L & W & E & R & b & t & l & r \\ \frac{1}{n} & \frac{1}{n} & 1-\frac{3}{n} & \frac{1}{n} & \frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} & 1-\frac{1}{n} \end{array}\right), \mu_n = \begin{pmatrix} x_2 & x_3 \\ \frac{\frac{1}{n}}{\frac{1}{n}+1-\frac{3}{n}} & \frac{1-\frac{3}{n}}{\frac{1}{n}+1-\frac{3}{n}} \end{pmatrix}.$$

Clearly $\lim_{n\to\infty}\sigma_n = \sigma$ and $\lim_{n\to\infty}\mu_n = \mu$. □

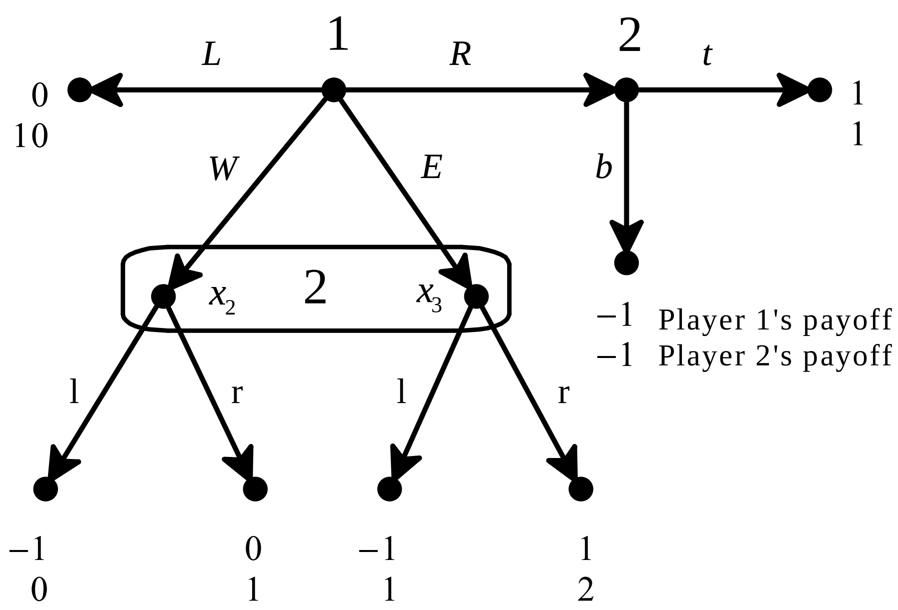

Figure 12.12: The extensive form for Exercise 12.2.

Player 2

Player 1		*tl*	*tr*	*bl*	*br*
	L	0 , 10	0 , 10	0 , 10	0 , 10
	R	1 , 1	1 , 1	−1, −1	−1, −1
	W	−1, 0	0 , 1	−1, 0	0 , 1
	E	−1, 1	1 , 2	−1, 1	1 , 2

Figure 12.13: The strategic-form of the game of Figure 12.12.

Solutions to Exercise 12.3 The game under consideration is shown in Figure 12.14.

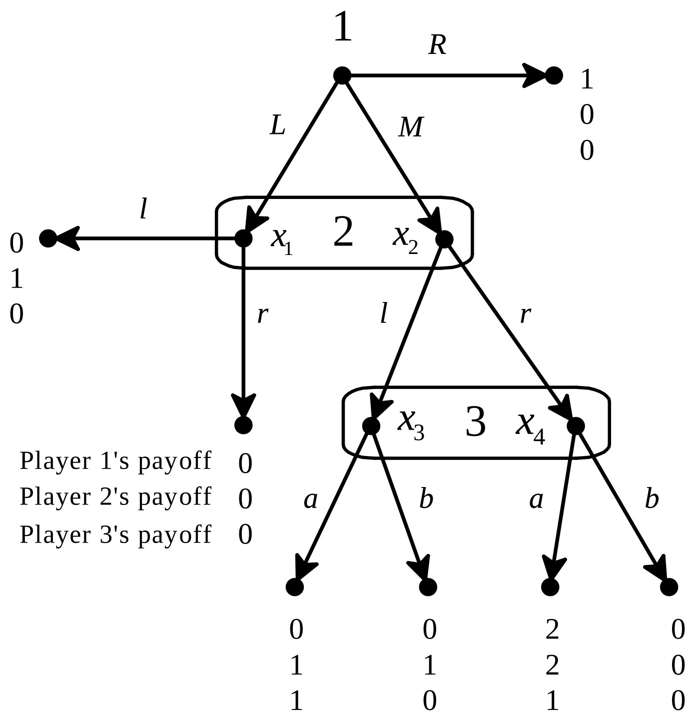

Figure 12.14: The extensive form for Exercise 12.3.

(a) The corresponding strategic form is shown in Figure 12.15.

Player 3 chooses *a*:

Player 1 \ Player 2	l	r
R	**1 , 0 , 0**	1 , 0 , 0
M	0 , 1 , 1	**2 , 2 , 1**
L	0 , 1 , 0	0 , 0 , 0

Player 3 chooses *b*:

Player 1 \ Player 2	l	r
R	**1 , 0 , 0**	**1 , 0 , 0**
M	0 , 1 , 0	0 , 0 , 0
L	0 , 1 , 0	0 , 0 , 0

Figure 12.15: The strategic-form of the game of Figure 12.14.

(b) The pure-strategy Nash equilibria are: (R,l,a), (M,r,a), (R,l,b) and (R,r,b). They are all subgame perfect because there are no proper subgames. $(R,l,b))$ and (R,r,b) cannot be part of a sequential equilibrium because b is a strictly dominated choice at Player 3's information set and, therefore, (R,l,b) and (R,r,b) would violate sequential rationality (with any system of beliefs).
On the other hand, both (R,l,a) and (M,r,a) can be part of an assessment which is a sequential equilibrium.

First we show that $\sigma = (R, l, a)$ together with the system of beliefs $\mu = \left(\begin{array}{cc|cc} x_1 & x_2 & x_3 & x_4 \\ \frac{2}{3} & \frac{1}{3} & 1 & 0 \end{array}\right)$ is a sequential equilibrium.

Consider the sequence of completely mixed strategy profiles whose n^{th} term is $\sigma_n = \left(\begin{array}{ccc|cc|cc} L & M & R & l & r & a & b \\ \frac{2}{n} & \frac{1}{n} & 1-\frac{3}{n} & 1-\frac{1}{n} & \frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} \end{array}\right)$. Clearly $\lim_{n\to\infty} \sigma_n = \sigma$.

The corresponding Bayesian system of beliefs has n^{th} term

$$\mu_n = \left(\begin{array}{cc|cc} x_1 & x_2 & x_3 & x_4 \\ \frac{\frac{2}{n}}{\frac{2}{n}+\frac{1}{n}} = \frac{2}{3} & \frac{\frac{1}{n}}{\frac{2}{n}+\frac{1}{n}} = \frac{1}{3} & \frac{\frac{1}{n}\left(1-\frac{1}{n}\right)}{\frac{1}{n}\left(1-\frac{1}{n}\right)+\frac{1}{n}\left(\frac{1}{n}\right)} = 1-\frac{1}{n} & \frac{\frac{1}{n}\left(\frac{1}{n}\right)}{\frac{1}{n}\left(1-\frac{1}{n}\right)+\frac{1}{n}\left(\frac{1}{n}\right)} = \frac{1}{n} \end{array}\right)$$

Clearly $\lim_{n\to\infty} \mu_n = \mu$. Thus the assessment is consistent (Definition 12.1.1).

Sequential rationality is easily checked: given the strategy profile and the system of beliefs,

(1) for Player 3, a yields 1, while b yields 0,

(2) for Player 2, l yields 1, while r yields $\frac{2}{3}(0) + \frac{1}{3}(2) = \frac{2}{3}$,

(3) for Player 1, R yields 1, while L and M yield 0.

Next we show that $\sigma = (M, r, a)$ together with $\mu = \left(\begin{array}{cc|cc} x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & 0 & 1 \end{array}\right)$ is a sequential equilibrium.

Consider the sequence of completely mixed strategy profiles whose n^{th} term is

$$\sigma_n = \left(\begin{array}{ccc|cc|cc} L & M & R & l & r & a & b \\ \frac{1}{2n} & 1-\frac{1}{n} & \frac{1}{2n} & \frac{1}{n} & 1-\frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} \end{array}\right).$$

Clearly $\lim_{n\to\infty} \sigma_n = \sigma$. The corresponding Bayesian system of beliefs has n^{th} term

$$\mu_n = \left(\begin{array}{cc|cc} x_1 & x_2 & x_3 & x_4 \\ \frac{\frac{1}{2n}}{\frac{1}{2n}+\left(1-\frac{1}{n}\right)} = \frac{1}{2n-1} & 1-\frac{1}{2n-1} & \frac{\frac{1}{n}\left(1-\frac{1}{n}\right)}{\frac{1}{n}\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)^2} = \frac{1}{n} & 1-\frac{1}{n} \end{array}\right)$$

Clearly $\lim_{n\to\infty} \mu_n = \mu$. Thus the assessment is consistent (Definition 12.1.1). Sequential rationality is easily checked: given the strategy profile and the system of beliefs,

(1) for Player 3, a yields 1, while b yields 0,

(2) for Player 2, l yields 1, while r yields 2 and

(3) for Player 1, M yields 2, while R yields 1 and L yields 0.

□

Solutions to Exercise 12.4 The game under consideration is shown in Figure 12.16.

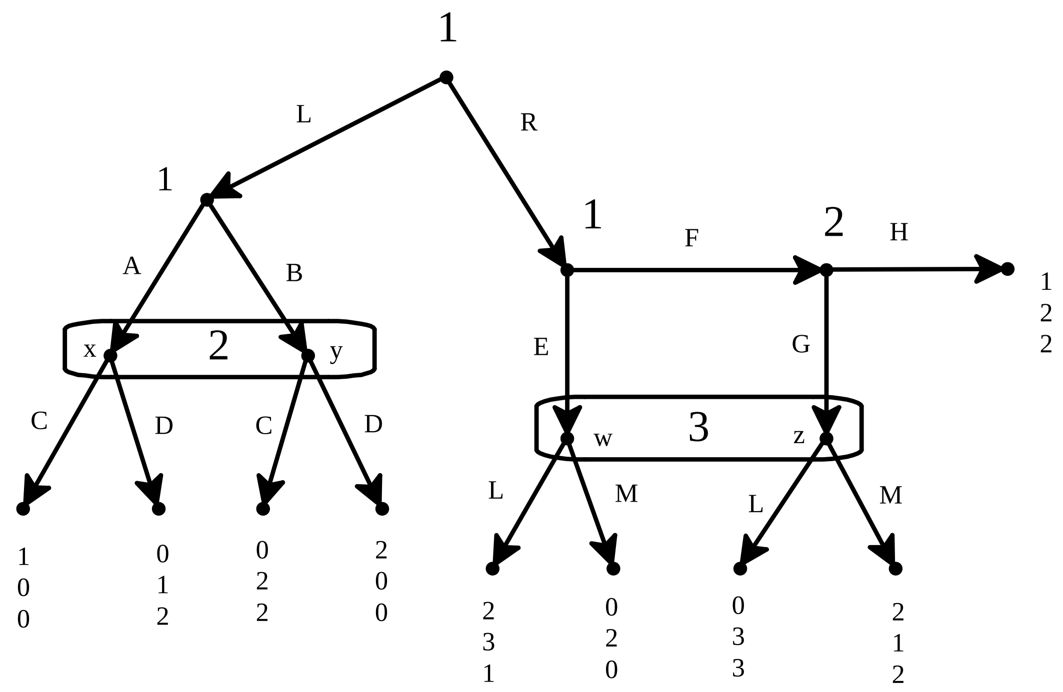

Figure 12.16: The extensive-from game for Exercise 12.4.

(a) First we solve the subgame on the left, whose strategic form is shown in Figure 12.17.

		Player 2: *C*	Player 2: *D*
Player 1	*A*	1 , 0	0 , 1
	B	0 , 2	2 , 0

Figure 12.17: The strategic-form of the subgame that starts after choice *L* of Player 1.

There is no pure-strategy Nash equilibrium. Let p be the probability of A and q the probability of C.

Then at a Nash equilibrium it must be that $q = 2(1-q)$ and $2(1-p) = p$.

Thus there is a unique Nash equilibrium given by $\left(\begin{array}{cc|cc} A & B & C & D \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right)$ with an expected payoff of $\frac{2}{3}$ for both players.

Next consider the subgame on the right. In this subgame the following are pure-strategy Nash equilibria: (F,H,M) (where Player 1's payoff is 1), (E,L,H) (where Player 1's payoff is 2), and (E,L,G) (where Player 1's payoff is 2).

Thus the following are subgame-perfect equilibria:

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right),$$

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right).$$

(b) $\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$ cannot be part of a weak sequential equilibrium, because choice M is strictly dominated by L and thus there are no beliefs at Player 3's information set that justify choosing M.

$\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$ cannot be part of a weak sequential equilibrium, because (given that Player 3 chooses L) H is not a sequentially rational choice for Player 2 at his singleton node.

$\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$ is a weak sequential equilibrium with the system of beliefs $\left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{1}{3} & 1 & 0 \end{array}\right)$.

(c) $\left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$ together with beliefs $\mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{2}{3} & 1 & 0 \end{array}\right)$ is a sequential equilibrium.

Consistency can be verified with the sequence of completely mixed strategies

$$\sigma_n = \left(\begin{array}{cc|cc|cc|cc|cc|cc} L & R & A & B & C & D & E & F & G & H & L & M \\ \frac{1}{n} & 1-\frac{1}{n} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 1-\frac{1}{n} & \frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} & 1-\frac{1}{n} & \frac{1}{n} \end{array}\right).$$

For example,

$$\mu_n(w) = \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)\frac{1}{n}\left(1-\frac{1}{n}\right)} = \frac{1}{1+\frac{1}{n}} \to 1 \text{ as } n \to \infty.$$

Sequential rationality is easily verified. □

Solutions to Exercise 12.5

(a) The extensive form is shown in Figure 12.18, where FF means 1Off-2Off, FN means 1Off-2On, etc.

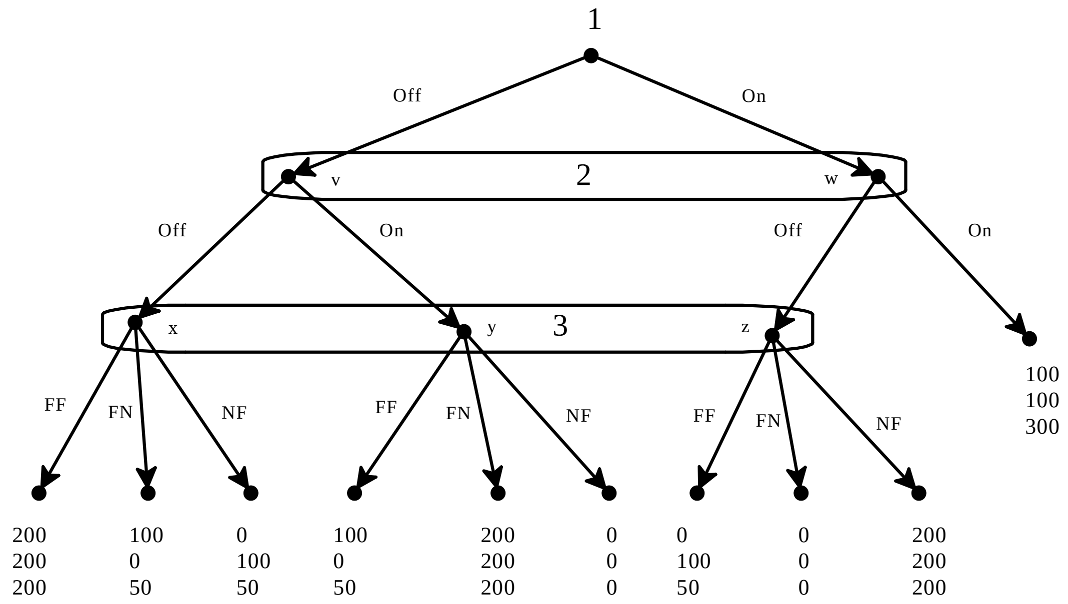

Figure 12.18: The extensive form for Exercise 12.5.

(b) The corresponding strategic form is shown in Figure 12.19.

Player 2

Player 1		*On*	*Off*
	On	100 , 100 , 300	200 , 200 , 200
	Off	0 , 0 , 0	0 , 100 , 50

Player 3: 1On-2Off

Player 2

Player 1		*On*	*Off*
	On	100 , 100 , 300	0 , 0 , 0
	Off	200 , 200 , 200	100 , 0 , 50

Player 3: 1Off-2On

Player 2

Player 1		*On*	*Off*
	On	100 , 100 , 300	0 , 100 , 100
	Off	100 , 0 , 50	200 , 200 , 200

Player 3: both Off

Figure 12.19: The strategic form of the game of Figure 12.18.

(c) The Nash equilibria are highlighted in the strategic form: (On, Off, 1On-2Off), (Off, On, 1Off-2On), (On, On, both-Off) and (Off, Off, both-Off).

(d) (On, On, both-Off) cannot be part of a sequential equilibrium. First of all, for Player 3 'both-Off' is a sequentially rational choice only if Player 3 attaches (sufficiently high) positive probability to node x.

However, consistency does not allow beliefs with $\mu(x) > 0$. To see this, consider a sequence of completely mixed strategies $\{p_n, q_n\}$ for Players 1 and 2, where p_n is the probability with which Player 1 chooses Off and q_n is the probability with which Player 2 chooses Off and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 0$.

Then, by Bayesian updating,

$$P_n(x|\{x,y,z\}) = \frac{p_n q_n}{p_n q_n + p_n(1-q_n) + q_n(1-p_n)}. \qquad (\blacklozenge)$$

If q_n goes to 0 as fast as, or faster than, p_n (that is, if

$$\lim_{n\to\infty} \frac{q_n}{p_n}$$

is finite), then divide numerator and denominator of $(\blacklozenge)$ by p_n to get

$$P_n(x|\{x,y,z\}) = \frac{q_n}{q_n + (1-q_n) + \frac{q_n}{p_n}(1-p_n)}$$

Taking the limit as $n \to \infty$ we get

$$\frac{0}{0+1+\left(\lim_{n\to\infty} \frac{q_n}{p_n}\right)(1)} = 0.$$

[If p_n goes to 0 as fast as or faster than q_n then repeat the above argument by dividing by q_n.] Thus a consistent assessment must assign zero probability to node x.

□

Solutions to Exercise 12.6

(a) The extensive-form game is shown in Figure 12.20.

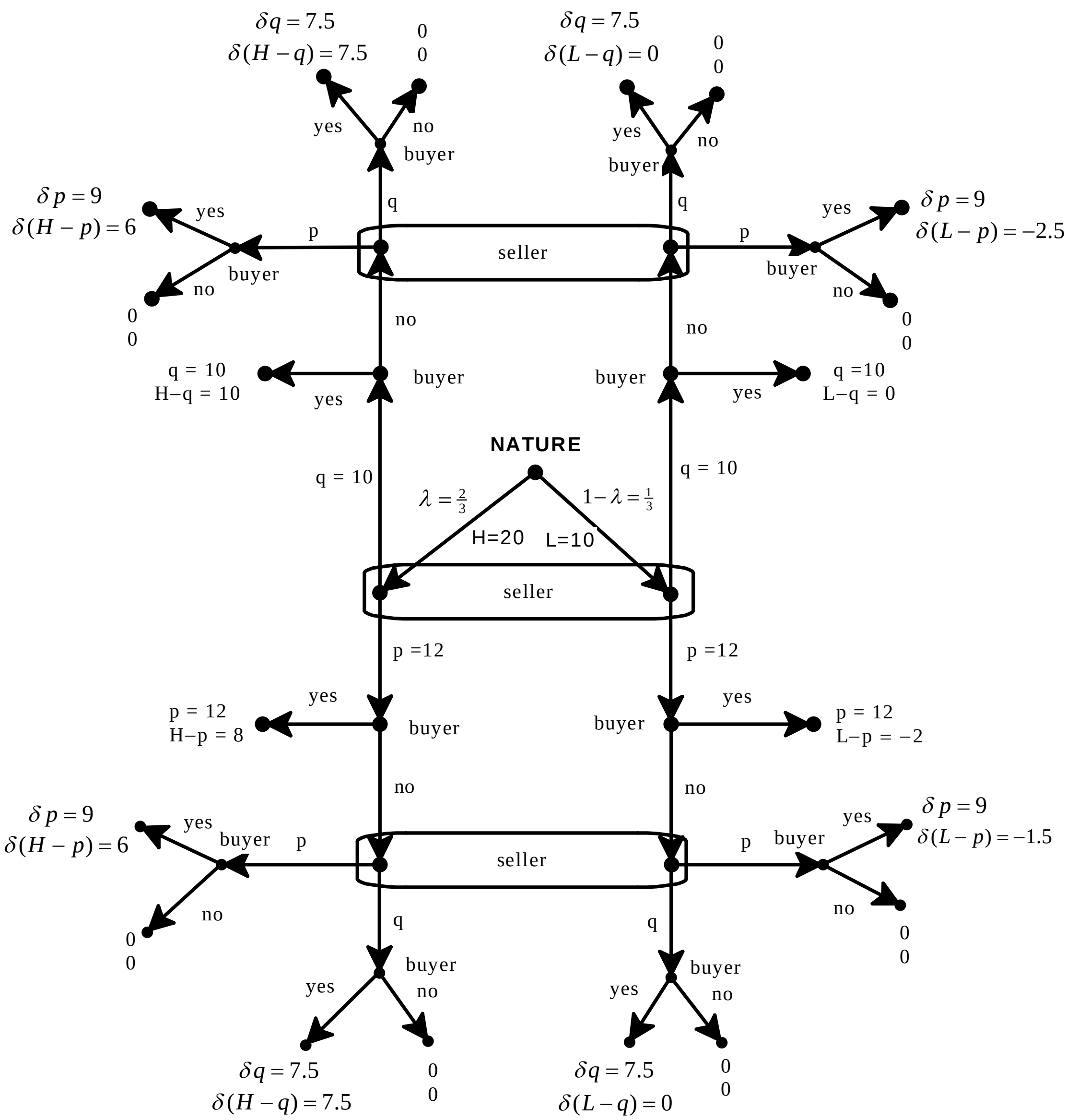

Figure 12.20: The buyer-seller game of Exercise 12.6.

(b) The following assessment (σ, μ) is a sequential equilibrium. The pure-strategy profile σ is as follows:

The seller offers $p = 12$ in period 1 and, if his offer is rejected, adjusts his offer to $q = 10$ in period 2; furthermore, if the first-period offer had been $q = 10$ and it had been rejected then he would have offered $q = 10$ again in the second period.

The H **buyer** (that is, the buyer at information sets that follow Nature's choice of H) always says Yes to any offer of the seller.

The L **buyer** (that is, the buyer at information sets that follow Nature's choice of L) always says No to an offer of p and always says Yes to an offer of q. The system

of beliefs is as follows (where TL means the left node of the top information set of the seller, TR the right node of that information set, ML means the left node of the middle information set of the seller, MR the right node of that information set, BL means the left node of the bottom information set of the seller, BR the right node of that information set):

$$\mu = \left(\begin{array}{cc|cc|cc} TL & TR & ML & MR & BL & BR \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 1 \end{array} \right).$$

Let us first check sequential rationality. The seller's payoff is (recall that $\delta = \frac{3}{4}$)

$$\left(\tfrac{2}{3}\right) 12 + \left(\tfrac{1}{3}\right) \left(\tfrac{3}{4}\right) 10 = \tfrac{63}{6}$$

which is greater than the payoff he would get if he offered $q = 10$, namely a payoff of 10.

The H-type's payoff is $20 - 12 = 8$, while if she said No to $p = 12$ and then Yes to $q = 10$ in period 2 her payoff would be $\frac{3}{4}(20 - 10) = \frac{15}{2} = 7.5$.

Furthermore, for the H type, at every node of hers, saying Yes is always strictly better than saying No.

The L-type's payoff is 0, while if she said Yes to $p = 12$ then her payoff would be -2.

At every node of the L type after having been offered $p = 12$ saying No is strictly better than saying Yes and at every node after having been offered $q = 10$ saying No gives the same payoff as saying Yes, namely 0.

To check consistency, construct the following completely mixed strategy profile $\langle \sigma_n \rangle_{n=1,2,\ldots}$:

(1) for the seller and for the L-buyer, any choice that has zero probability in σ is assigned probability $\frac{1}{n}$ in σ_n and any choice that has probability 1 in σ is assigned probability $1 - \frac{1}{n}$ in σ_n,

(2) for the H-buyer, any choice that has zero probability in σ is assigned probability $\frac{1}{n^2}$ in σ_n and any choice that has probability 1 in σ is assigned probability $1 - \frac{1}{n^2}$ in σ_n.

Let us compute the corresponding beliefs μ_n at the top and at the bottom information sets of the seller:

$$\mu_n(TL) = \frac{\frac{1}{n}\left(\frac{1}{n^2}\right)}{\frac{1}{n}\left(\frac{1}{n^2}\right) + \frac{1}{n}\left(\frac{1}{n}\right)} = \frac{1}{1+n} \quad \text{and} \quad \mu_n(TR) = \frac{\frac{1}{n}\left(\frac{1}{n}\right)}{\frac{1}{n}\left(\frac{1}{n^2}\right) + \frac{1}{n}\left(\frac{1}{n}\right)} = \frac{1}{\frac{1}{n}+1}.$$

Thus $\lim_{n\to\infty} \mu_n(TL) = 0 = \mu(TL)$ and $\lim_{n\to\infty} \mu_n(TR) = 1 = \mu(TR)$.

$$\mu_n(BL) = \frac{\left(1-\frac{1}{n}\right)\left(\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)\left(\frac{1}{n}\right) + \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)} = \frac{1}{n}$$

and $\mu_n(BR) == \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)\left(\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)} = 1 - \frac{1}{n};$

thus $\lim_{n\to\infty} \mu_n(BL) = 0 = \mu(BL)$ and $\lim_{n\to\infty} \mu_n(BR) = 1 = \mu(BR)$.

□