13. Perfect Bayesian Equilibrium

13.1 Belief revision and AGM consistency

Any attempt to refine the notion of subgame-perfect equilibrium in extensive-form (or dynamic) games must deal with the issue of *belief revision*: how should a player revise her beliefs when informed that she has to make a choice at an information set of hers to which she initially assigned zero probability? As we saw in the previous chapter, Kreps and Wilson (1982) suggested the notion of a consistent assessment (Definition 12.1.1, Chapter 12) to deal with this issue. From now on, we shall refer to the notion of consistency proposed by Kreps and Wilson, as *KW-consistency* (KW stands for 'Kreps-Wilson'), in order to distinguish it from a different notion of consistency, called *AGM-consistency*, that will be introduced in this section. We shall make use of concepts developed in Section 9.4 (Chapter 9): the reader might want to review that material before continuing.

In this chapter it will be more convenient to use the so called "history-based" definition of extensive-form game, which is spelled out in Section 13.6. Essentially it boils down to identifying a node in the tree with the sequence of actions leading from the root to it. We call a sequence of actions starting from the root of the tree a *history*.

For example, consider the extensive form of Figure 13.1.

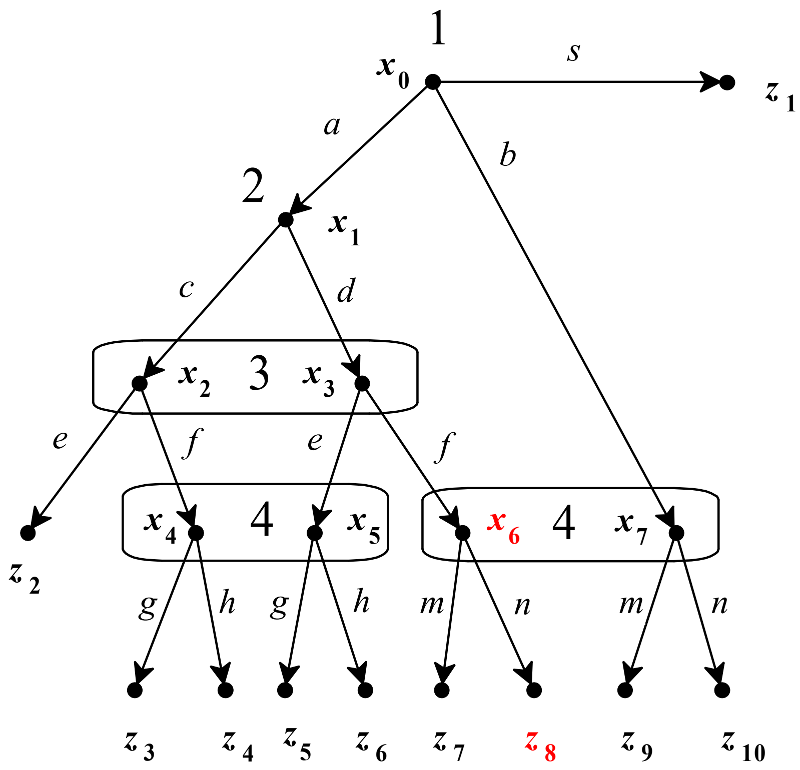


Figure 13.1: A game-frame in extensive form.

Node x_0 (the root of the tree) is identified with the null or empty history $\emptyset$, decision node x_6 with the history (or sequence of actions) adf, terminal node z_8 with history $adfn$, etc. Thus it will no longer be necessary to label the nodes of the tree, because we can refer to them by naming the corresponding histories.

If h is a decision history we denote by $A(h)$ the set of actions (or choices) available at h. For example, in the game of Figure 13.1, $A(a) = \{c,d\}$, $A(ac) = \{e, f\}$, $A(adf) = \{m,n\}$, etc. If h is a history and a is an action available at h (that is $a \in A(h)$), then we denote by ha the history obtained by appending a to h.

Definition 13.1.1 Given a set H, a *total pre-order* on H is a binary relation $\succsim \subseteq H \times H$ which is complete ($\forall h, h' \in H$, either $h \succsim h'$ or $h' \succsim h$, or both) and transitive ($\forall h, h', h'' \in H$, if $h \succsim h'$ and $h' \succsim h''$ then $h \succsim h''$).
We write
$h \sim h'$ as a short-hand for "$h \succsim h'$ and $h' \succsim h$", and
$h \succ h'$ as a short-hand for "$h \succsim h'$ and $h' \not\succsim h$".

We saw in Chapter 9 (Section 9.4.2) that the AGM theory of belief revision (introduced by Alchourrón *et al.*, 1985) is intimately linked to the notion of a plausibility order. This is reflected in the following definition.

Definition 13.1.2 Given an extensive form, a *plausibility order* is a total pre-order on the set of histories H that satisfies the following properties (D denotes the set of decision histories, $A(h)$ the set of actions available at decision history h and $I(h)$ the information set that contains decision history h): for all $h \in D$,

$PL1.$ $h \succsim ha$, for all $a \in A(h)$.

$PL2.$ (i) There exists an $a \in A(h)$ such that $ha \sim h$,
(ii) for all $a \in A(h)$, if $ha \sim h$ then $h'a \sim h'$, for all $h' \in I(h)$.

$PL3.$ If history h is assigned to chance (or Nature), then $ha \sim h$, for all $a \in A(h)$.

The interpretation of $h \succsim h'$ is that history h is *at least as plausible as* history h' (thus the interpretation of $h \sim h'$ is that h is *just as plausible* as h' and the interpretation of $h \succ h'$ is that h is *more plausible* than h').

- Property *PL1* of Definition 13.1.2 says that adding an action to a decision history h cannot yield a more plausible history than h itself.
- Property *PL2* says that at every decision history h there is at least one action a which is "plausibility preserving" in the sense that adding a to h yields a history which is as plausible as h; furthermore, any such action a performs the same role with any other history that belongs to the same information set as h.
- Property *PL3* says that all the actions at a history assigned to chance are plausibility preserving.

Definition 13.1.3 Given an extensive-form, an assessment (σ, μ) (see Definition 11.1.1, Chapter 11) is *AGM-consistent* if there exists a plausibility order $\succsim$ on H such that:
(i) the actions that are assigned positive probability by σ are precisely the plausibility-preserving actions: for all $h \in D$ and for all $a \in A(h)$,

$$\sigma(a) > 0 \text{ if and only if } h \sim ha, \tag{P1}$$

(ii) the histories that are assigned positive probability by μ are precisely those that are most plausible within the corresponding information set: for all $h \in D$,

$$\mu(h) > 0 \text{ if and only if } h \succsim h', \text{ for all } h' \in I(h). \tag{P2}$$

If $\succsim$ satisfies properties $(P1)$ and $(P2)$ with respect to (σ, μ), we say that $\succsim$ *rationalizes* (σ, μ).

In conjunction with sequential rationality, the notion of AGM-consistency is sufficient to rule out some subgame-perfect equilibria. Consider, for example, the extensive game of Figure 13.2 and the pure-strategy profile $\sigma = (c,d,f)$ (highlighted by double edges), which constitutes a Nash equilibrium of the game (and also a subgame-perfect equilibrium since there are no proper subgames).

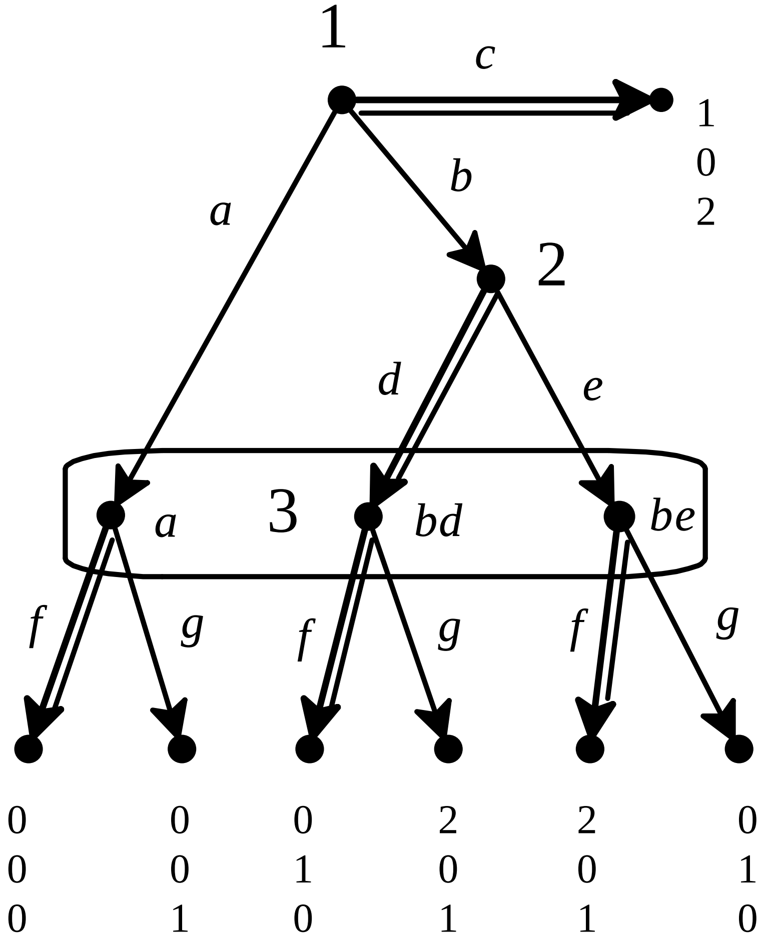


Figure 13.2: The strategy profile (c,d,f) cannot be part of an AGM-consistent assessment

Can $\sigma = (c,d,f)$ be part of a sequentially rational AGM-consistent assessment (σ,μ)? Since, for Player 3, choice f is rational only if the player assigns (sufficiently high) positive probability to history *be* (at histories *a* and *bd*, g yields a higher payoff than f for Player 3), sequential rationality requires that $\mu(be) > 0$; however, any such assessment is *not* AGM-consistent.

In fact, if there were a plausibility order $\succsim$ that satisfied Definition 13.1.3, then, by *P1*, $b \sim bd$ (since $\sigma(d) = 1 > 0$) and $b \succ be$ (since $\sigma(e) = 0$) and, by *P2*,

$$be \succsim bd \tag{13.1}$$

(since – by hypothesis – μ assigns positive probability to *be*). By transitivity of $\succsim$, from $b \succ be$ and $b \sim bd$ it follows that $bd \succ be$, contradicting (13.1).

On the other hand, the Nash equilibrium

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right)$$

together with the system of beliefs

$$\mu = \begin{pmatrix} a & bd & be \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

forms a sequentially rational, AGM-consistent assessment: it can be rationalized by the following plausibility order (we use the convention that if the row to which history h belongs is above the row to which history h' belongs, then $h \succ h'$, that is, h is more plausible than h', and if h and h' belong to the same row then $h \sim h'$, that is, h and h' are equally plausible; as usual, $\emptyset$ denotes the null history, that is, the root of the tree):

$$\begin{pmatrix} \text{most plausible} & \emptyset, c \\ & b, bd, be, bdf, bdg, bef, beg \\ \text{least plausible} & a, af, ag \end{pmatrix}$$

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 13.8.1 at the end of this chapter.

13.2 Bayesian consistency

The definition of AGM-consistency pertains to the supports of a given assessment, that is, with the actions that are assigned positive probability by the strategy profile σ and the histories that are assigned positive probability by the system of beliefs μ. In this sense it is a *qualitative* (as opposed to quantitative) property: *how the probabilities are distributed on those supports is irrelevant for AGM-consistency*. However, AGM-consistency is not sufficient: we also need to impose quantitative restrictions concerning the actual probabilities. The reason for this is that we want the given assessment to satisfy "Bayesian updating as long as possible". By this we mean the following:

1. when information causes no surprises, because the play of the game is consistent with the most plausible play(s) (that is, when information sets are reached that have positive prior probability), then beliefs should be formed using Bayesian updating (Definition 9.4.1, Chapter 9), and
2. when information is surprising (that is, when an information set is reached that had zero prior probability) then new beliefs can be formed in an arbitrary way, but from then on Bayesian updating should be used to update those new beliefs, whenever further information is received that is consistent with those new beliefs.

The next definition formalizes the above requirements.

Definition 13.2.1 Given a finite extensive form, let $\succsim$ be a plausibility order that rationalizes the assessment (σ,μ). We say that (σ,μ) is *Bayesian* (or *Bayes*) *consistent* relative to $\succsim$ if for every equivalence class E of $\succsim$ that contains some decision history h with $\mu(h)>0$ $\left(\text{that is, } E\cap D_\mu^+\neq\emptyset, \text{ where } D_\mu^+=\{h\in D:\mu(h)>0\}\right)$ there exists a probability distribution $\nu_E: H\to[0,1]$ (recall that H is a finite set) such that:

B1. $\nu_E(h)>0$ if and only if $h\in E\cap D_\mu^+$.

B2. If $h,h'\in E\cap D_\mu^+$ and $h'=ha_1\ldots a_m$ (that is, h is a prefix of h') then $\nu_E(h')=\nu_E(h)\times\sigma(a_1)\times\cdots\times\sigma(a_m)$.

B3. If $h\in E\cap D_\mu^+$ then, for every $h'\in I(h)$, $\quad \mu(h')=\nu_E\left(h'|I(h)\right)\overset{def}{=}\dfrac{\nu_E(h')}{\sum\limits_{h''\in I(h)}\nu_E(h'')}$.

Property $B1$ requires that $\nu_E(h)>0$ if and only if $h\in E$ and $\mu(h)>0$.[1]

Property $B2$ requires ν_E to be consistent with the strategy profile σ in the sense that if $h,h'\in E$, $\nu_E(h)>0$, $\nu_E(h')>0$ and $h'=ha_1...a_m$ then the probability of h' (according to ν_E) is equal to the probability of h multiplied by the probabilities (according to σ) of the actions that lead from h to h'.[2]

Property $B3$ requires the system of beliefs μ to satisfy Bayes' rule in the sense that if $h\in E$ and $\mu(h)>0$ (so that E is the equivalence class of the most plausible elements of the information set $I(h)$) then, for every history $h'\in I(h)$, $\mu(h')$ (the probability assigned to h' by μ) coincides with the probability of h' conditional on $I(h)$, using the probability measure ν_E.

For an example of an AGM-consistent and Bayesian-consistent assessment, consider the extensive form of Figure 13.3. The assessment

$$\sigma=\left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g\\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{2}{3}\end{array}\right)\qquad \mu=\left(\begin{array}{ccc|ccc} ad & ae & b & a & bf & bg\\ \frac{1}{4} & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{2}{4}\end{array}\right)$$

is AGM-consistent because it is rationalized by the following plausibility order:

$$\left(\begin{array}{ll} \text{most plausible} & \emptyset, c\\ & a,b,ad,bf,bg,adf,adg,bfd,bgd\\ \text{least plausible} & ae,aef,aeg,bfe,bge\end{array}\right)$$

Furthermore, it is Bayesian relative to this plausibility order. First of all, note that $D_\mu^+=\{\emptyset,a,ad,b,bf,bg\}$.[3] Let $E=\{\emptyset,c\}$ be the top equivalence class, $F=\{a,b,ad,bf,bg,adf,adg,bfd,bgd\}$ the middle one and $G=\{ae,aef,aeg,bfe,bge\}$ the bottom one. Then only E and F have a non-empty intersection with D_μ^+ and thus, by Definition 13.2.1, we only need to specify two probability distributions: ν_E and ν_F. The first one is trivial: since $D_\mu^+\cap E=\{\emptyset\}$, it must be $\nu_E(\emptyset)=1$ (and $\nu_E(c)=0$). Since $D_\mu^+\cap F=\{a,ad,b,bf,bg\}$, by B1 of Definition 13.2.1 it must be that $\nu_F(h)>0$ if and only if $h\in\{a,ad,b,bf,bg\}$.

[1] Thus $\nu_E(h)=0$ if and only if either $h\in H\backslash E$ or $\mu(h)=0$

[2] Note that if $h,h'\in E$ and $h'=ha_1...a_m$, then $\sigma(a_j)>0$, for all $j=1,...,m$. In fact, since $h'\sim h$, every action a_j is plausibility preserving and therefore, by Property $(P1)$ of Definition 13.1.3, $\sigma(a_j)>0$.

[3] Recall that $D_\mu^+\overset{def}{=}\{h\in D:\mu(h)>0\}$.

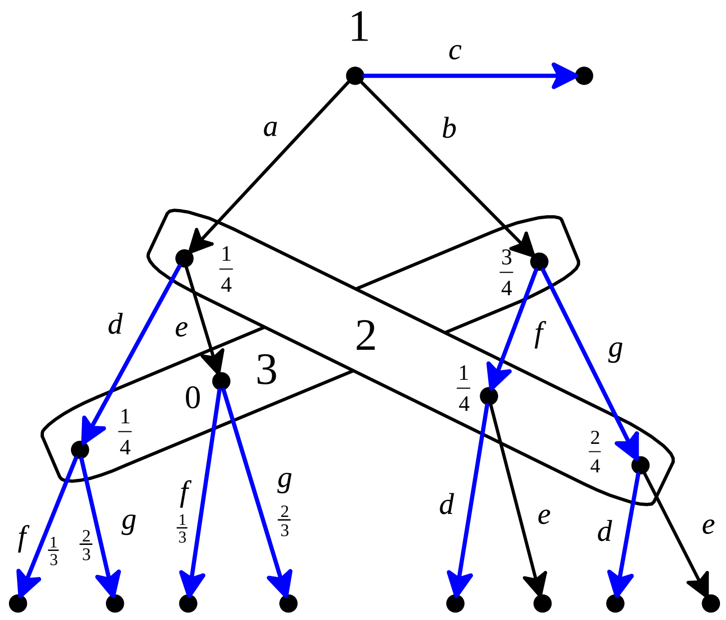


Figure 13.3: The assessment $\left(c,d,(\frac{1}{3}f,\frac{2}{3}g)\right)$, $\mu(ad)=\frac{1}{4}$, $\mu(b)=\frac{3}{4}$, $\mu(a)=\mu(bf)=\frac{1}{4}$, $\mu(bg)=\frac{2}{4}$ is both AGM-consistent and Bayes-consistent.

Consider the following probability distribution:

$$\nu_F = \begin{pmatrix} a & ad & b & bf & bg \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & \frac{2}{8} \end{pmatrix} \quad \text{(and } \nu_F(h)=0 \text{ for every other history } h\text{).}$$

Property B2 of Definition 13.2.1 is satisfied, because

$$\nu_F(ad) = \underbrace{\frac{1}{8}}_{=\nu_F(a)} \times \underbrace{1}_{\sigma(d)}, \quad \nu_F(bf) = \underbrace{\frac{3}{8}}_{=\nu_F(b)} \times \underbrace{\frac{1}{3}}_{\sigma(f)} \text{ and } \nu_F(bg) = \underbrace{\frac{3}{8}}_{=\nu_F(b)} \times \underbrace{\frac{2}{3}}_{\sigma(g)}.$$

To check that Property B3 of Definition 13.2.1 is satisfied, let $I_2 = \{a, bf, bg\}$ be the information set of Player 2 and $I_3 = \{b, ad, ae\}$ be the information set of Player 3. Then

$$\nu_F(I_2) = \nu_F(a) + \nu_F(bf) + \nu_F(bg) = \frac{4}{8} \quad \text{and}$$

$$\nu_F(I_3) = \nu_F(b) + \nu_F(ad) + \nu_F(ae) = \frac{3}{8} + \frac{1}{8} + 0 = \frac{4}{8}.$$

Thus

$$\frac{\nu_F(a)}{\nu_F(I_2)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4} = \mu(a), \quad \frac{\nu_F(bf)}{\nu_F(I_2)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4} = \mu(bf), \quad \frac{\nu_F(bg)}{\nu_F(I_2)} = \frac{\frac{2}{8}}{\frac{4}{8}} = \frac{2}{4} = \mu(bg)$$

$$\frac{\nu_F(b)}{\nu_F(I_3)} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4} = \mu(b), \quad \frac{\nu_F(ad)}{\nu_F(I_3)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4} = \mu(ad), \quad \frac{\nu_F(ae)}{\nu_F(I_3)} = \frac{0}{\frac{4}{8}} = 0 = \mu(ae).$$

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 13.8.2 at the end of this chapter.

13.3 Perfect Bayesian equilibrium

By adding sequential rationality (Definition 11.1.2, Chapter 11) to AGM-consistency (Definition 13.1.3) and Bayesian consistency (Definition 13.2.1) we obtain a new notion of equilibrium for extensive-form games.

Definition 13.3.1 Given a finite extensive-form game, an assessment (σ, μ) is a *perfect Bayesian equilibrium* (PBE) if it is sequentially rational, it is rationalized by a plausibility order on the set of histories and is Bayesian relative to that plausibility order.

For an example of a perfect Bayesian equilibrium, consider the game of Figure 13.4 (which is based on the game-frame of Figure 13.3) and the assessment considered in the previous section, namely

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array} \right) \quad \text{and} \quad \mu = \left(\begin{array}{ccc|ccc} ad & ae & b & a & bf & bg \\ \frac{1}{4} & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{array} \right).$$

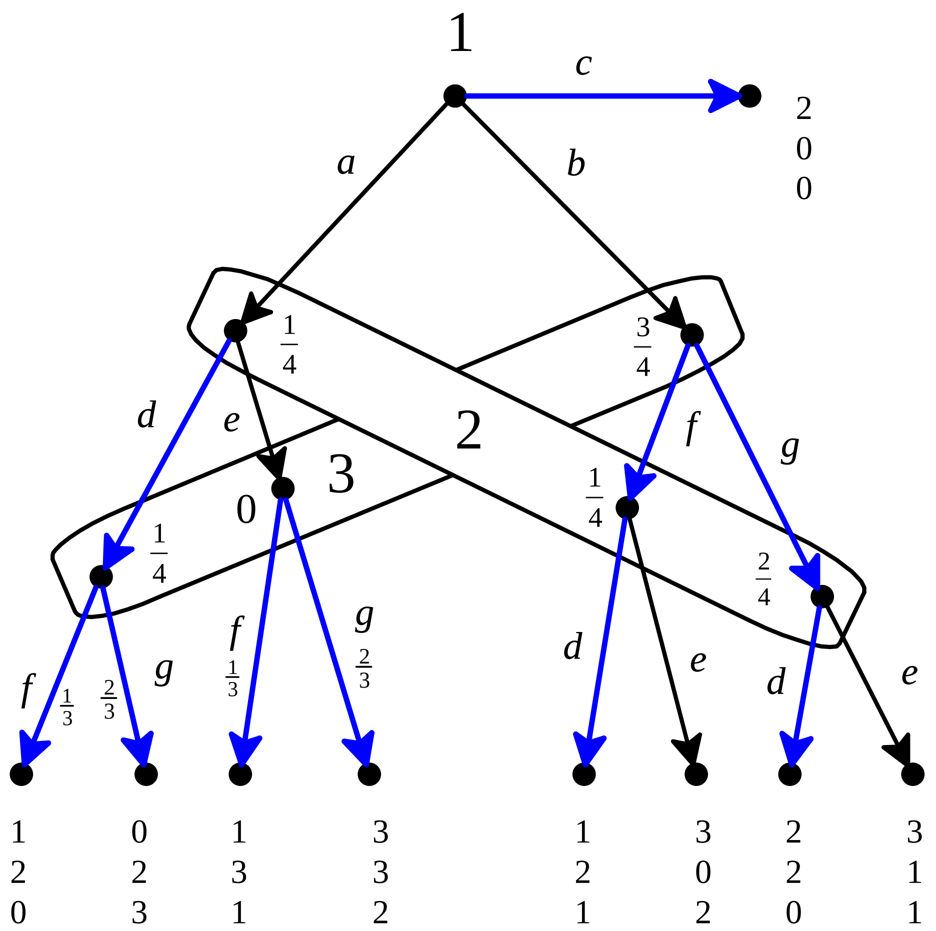


Figure 13.4: A game based on the game-frame of Figure 13.3.

We showed in the previous section that the assessment under consideration is AGM-consistent and Bayes-consistent. Thus we only need to verify sequential rationality.

For Player 3 the expected payoff from playing f is $\left(\frac{1}{4}\right)0+(0)1+\left(\frac{3}{4}\right)1=\frac{3}{4}$

and the expected payoff from playing g is $\left(\frac{1}{4}\right)3+(0)2+\left(\frac{3}{4}\right)0=\frac{3}{4}$;

thus any mixture of f and g is sequentially rational, in particular the mixture $\begin{pmatrix} f & g \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

For Player 2 the expected payoff from playing d is $\frac{1}{4}\left[\left(\frac{1}{3}\right)2+\left(\frac{2}{3}\right)2\right]+\left(\frac{1}{4}\right)2+\left(\frac{2}{4}\right)2=2$,

while the expected payoff from playing e is $\frac{1}{4}\left[\left(\frac{1}{3}\right)3+\left(\frac{2}{3}\right)3\right]+\left(\frac{1}{4}\right)0+\left(\frac{2}{4}\right)1=\frac{5}{4}$;

thus d is sequentially rational.

For Player 1, c gives a payoff of 2 while a gives a payoff of $\frac{1}{3}$ and b a gives a payoff of $\frac{5}{3}$;

thus c is sequentially rational.

We now turn to the properties of Perfect Bayesian equilibria.

Theorem 13.3.1 — Bonanno, 2013. Consider a finite extensive-form game and an assessment (σ,μ). If (σ,μ) is a perfect Bayesian equilibrium then

(1) σ is a subgame-perfect equilibrium and

(2) (σ,μ) is a weak sequential equilibrium.

The example of Figure 13.2 showed that not every subgame-perfect equilibrium can be part of a perfect Bayesian equilibrium. Thus, by Theorem 13.3.1, the notion of perfect Bayesian equilibrium is a strict refinement of the notion of subgame-perfect equilibrium.

Theorem 13.3.2 — Bonanno, 2013. Consider a finite extensive-form game. If the assessment (σ,μ) is sequential equilibrium then it is a perfect Bayesian equilibrium.

The next example shows that the notion of sequential equilibrium is a strict refinement of perfect Bayesian equilibrium.

Not every perfect Bayesian equilibrium is a sequential equilibrium. To see this, consider the game of Figure 13.5.

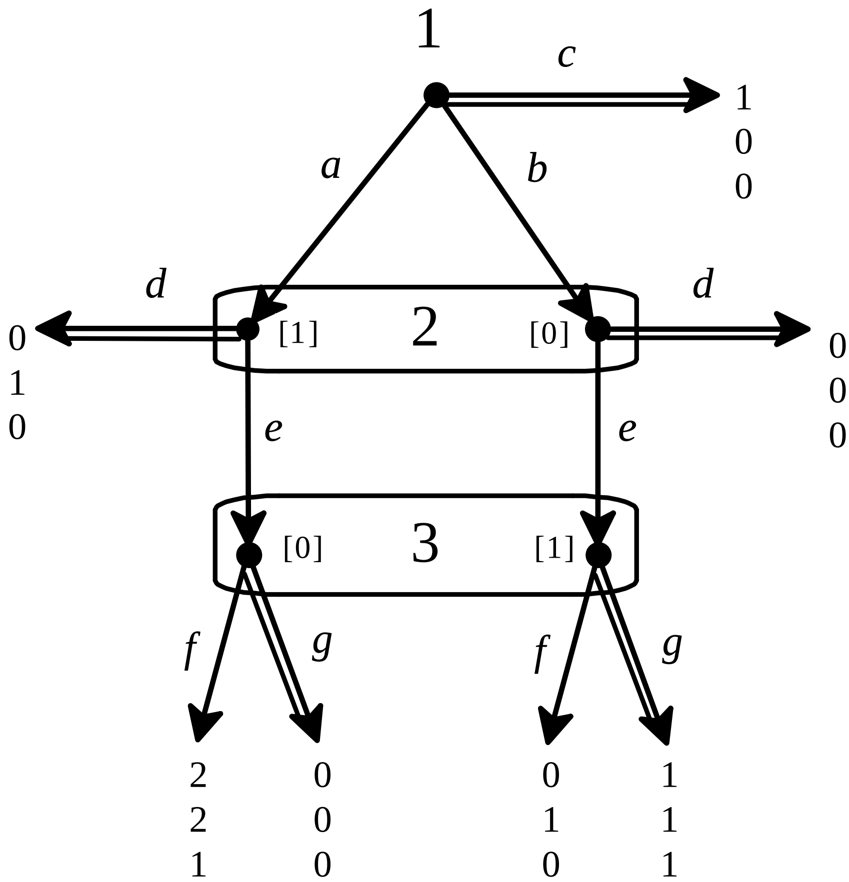


Figure 13.5: A game with a perfect Bayesian equilibrium which is not a sequential equilibrium.

A perfect Bayesian equilibrium of this game is given by the pure-strategy profile
$\sigma = (c, d, g)$ (highlighted by double edges),
together with the degenerate beliefs $\mu(a) = \mu(be) = 1$.
In fact, (σ, μ) is sequentially rational and, furthermore, it is rationalized by the plausibility order (▲) below and is Bayesian relative to it (the probability distributions on the equivalence classes that contain histories h with $\mu(h) > 0$ are written next to the order):

$$\begin{pmatrix} \text{most plausible} & \emptyset, c \\ & a, ad \\ & b, bd \\ & be, beg \\ & ae, aeg \\ & bef \\ \text{least plausible} & aef \end{pmatrix} \qquad \begin{pmatrix} \nu_{\{\emptyset,c\}}(\emptyset) = 1 \\ \nu_{\{a,ad\}}(a) = 1 \\ - \\ \nu_{\{be,beg\}}(be) = 1 \\ - \\ - \\ - \end{pmatrix} \qquad (\blacktriangle)$$

The belief revision policy encoded in a perfect Bayesian equilibrium can be interpreted either as the point of view of an external observer or as a belief revision policy which is shared by all the players. For example, the perfect Bayesian equilibrium under consideration (for the game of Figure 13.5), namely $\sigma = (c,d,g)$ and $\mu(a) = \mu(be) = 1$, reflects the following belief revision policy:

- the initial beliefs are that Player 1 will play c;
- conditional on learning that Player 1 did not play c, the observer would become convinced that Player 1 played a (that is, she would judge a to be strictly more plausible than b) and would expect Player 2 to play d;
- upon learning that Player 1 did not play c and Player 2 did not play d, the observer would become convinced that Player 1 played b and Player 2 played e, hence judging history be to be strictly more plausible than history ae, thereby reversing her earlier belief that a was strictly more plausible than b.

Such a belief revision policy is consistent with the AGM rationality axioms (Alchourrón *et al.*, 1985) but is incompatible with the notion of sequential equilibrium. In fact, (σ,μ) is *not* KW-consistent (Definition 12.1.1, Chapter 12). To see this, consider an arbitrary sequence $\langle\sigma_n\rangle_{n=1,2,\ldots}$ that converges to σ:

$$\sigma_n = \left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g \\ p_n & q_n & 1-p_n-q_n & 1-r_n & r_n & t_n & 1-t_n \end{array}\right)$$

with

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n = 0.$$

Then the corresponding sequence $\langle\mu_n\rangle_{n=1,2,\ldots}$ of beliefs obtained by Bayesian updating is given by

$$\mu_n = \left(\begin{array}{cc|cc} a & b & ae & be \\ \frac{p_n}{p_n+q_n} & \frac{q_n}{p_n+q_n} & \frac{p_n r_n}{p_n r_n+q_n r_n} = \frac{p_n}{p_n+q_n} & \frac{q_n r_n}{p_n r_n+q_n r_n} = \frac{q_n}{p_n+q_n} \end{array}\right).$$

Since $\mu_n(a) = \mu_n(ae)$, if $\lim_{n\to\infty} \mu_n(a) = \mu(a) = 1$ then $\lim_{n\to\infty} \mu_n(ae) = 1 \neq \mu(ae) = 0$.

By Theorem 12.2 (Chapter 12), every finite extensive-form game with cardinal payoffs has at least one sequential equilibrium and, by Theorem 13.3.2, every sequential equilibrium is a perfect Bayesian equilibrium. Thus the following theorem follows as a corollary of these two results.

Theorem 13.3.3 Every finite extensive-form game with cardinal payoffs has at least one perfect Bayesian equilibrium.

The relationship among the different notions of equilibrium introduced so far (Nash equilibrium, subgame-perfect equilibrium, weak sequential equilibrium, perfect Bayesian equilibrium and sequential equilibrium) is shown in the Venn diagram of Figure 13.6.

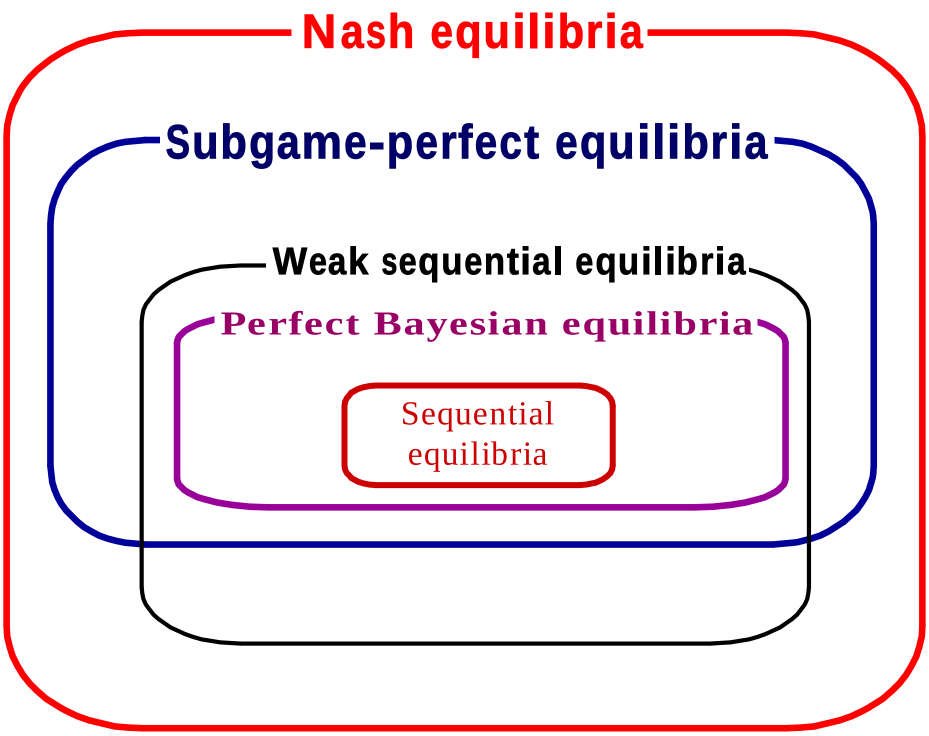


Figure 13.6: The relationship among Nash equilibria, subgame-perfect equilibria, weak sequential equilibria, perfect Bayesian equilibria and sequential equilibria.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 13.8.3 at the end of this chapter.

13.4 Adding independence

13.4.1 Weak independence

The notion of perfect Bayesian equilibrium imposes relatively mild restrictions on beliefs at information sets that are reached with zero probability. Those restrictions require consistency between the strategy profile σ and the system of belief μ, as well as the requirement of Bayesian updating "as long as possible" (that is, also after beliefs have been revised at unreached information sets). The example of Figure 13.5 showed that perfect Bayesian equilibrium is compatible with a belief revision policy that allows a reversal of judgment concerning the behavior of one player after observing an unexpected move by a different player. One might want to rule out such forms of belief revision. In this section we introduce and discuss further restrictions on belief revision that incorporate some form of independence.

"Questionable" belief revision policies like the one illustrated in the previous section[4] are ruled out by imposing the following restriction on the plausibility order:

if h and h' belong to the same information set $\big($that is, $h' \in I(h)\big)$
and a is an action available at h $\big(a \in A(h)\big)$, then

$$h \precsim h' \text{ if and only if } ha \precsim h'a. \qquad (IND_1)$$

(IND_1) says that if h is deemed to be at least as plausible as h' then the addition of any available action a must preserve this judgment, in the sense that ha must be deemed to be at least as plausible as $h'a$, and *vice versa*; it can also be viewed as an "independence" condition, in the sense that observation of a new action cannot lead to a change in the relative plausibility of previous histories.[5]

Any plausibility order that rationalizes the assessment $\sigma = (c,d,g)$ and $\mu(a) = \mu(be) = 1$ for the game of Figure 13.5 violates (IND_1), since it must be such that $a \succ b$ (because $\mu(a) > 0$ while $\mu(b) = 0$: see Definition 13.1.3) and also that $be \succ ae$ (because $\mu(be) > 0$ while $\mu(ae) = 0$: see Definition 13.1.3).

Property (IND_1) is a qualitative property (that is, a property that pertains to the plausibility order). We can add to it a quantitative condition on the probabilities to obtain a refinement of the notion of perfect Bayesian equilibrium. This quantitative property is given in Definition 13.4.1 and is a strengthening of the notion of Bayesian consistency introduced in Definition 13.2.1. First we need to define a "full-support common prior".

Let (σ,μ) be an assessment which is rationalized by a plausibility order $\precsim$. As before, let D^+_μ be the set of decision histories to which μ assigns positive probability: $D^+_\mu = \{h \in D : \mu(h) > 0\}$. Let $\mathcal{E}^+_\mu$ be the set of equivalence classes of $\precsim$ that have a non-empty intersection with D^+_μ. Clearly $\mathcal{E}^+_\mu$ is a non-empty, finite set. Suppose that (σ,μ) is Bayesian relative to $\precsim$ and let $\{\nu_E\}_{E \in \mathcal{E}^+_\mu}$ be a collection of probability density functions that satisfy the properties of Definition 13.2.1.
We call a probability density function $\nu : D \to (0,1]$ a *full-support common prior* of $\{\nu_E\}_{E \in \mathcal{E}^+_\mu}$ if, for every $E \in \mathcal{E}^+_\mu$, $\nu_E(\cdot) = \nu(\cdot \mid E \cap D^+_\mu)$, that is,

$$\text{for all } h \in E \cap D^+_\mu, \qquad \nu_E(h) = \frac{\nu(h)}{\sum\limits_{h' \in E \cap D^+_\mu} \nu(h')}.$$

Note that a full support common prior assigns positive probability to *all* decision histories, not only to those in D^+_μ.

---

[4]Reversal of relative likelihood judgments implied by the belief revision policy encoded in the assessment $\sigma = (c,d,g)$ and $\mu(a) = \mu(be) = 1$ for the game of Figure 13.5.

[5]Note, however, that (IND_1) is compatible with the following: $a \succ b$ $\big($with $b \in I(a)\big)$ and $bc \succ ad$ $\big($with $bc \in I(ad)$, $c,d \in A(a)$, $c \neq d\big)$.

Definition 13.4.1 Consider an extensive form. Let (σ,μ) be an assessment which is rationalized by the plausibility order $\precsim$ and is Bayesian relative to it and let $\{\nu_E\}_{E\in\mathscr{E}_\mu^+}$ be a collection of probability density functions that satisfy the properties of Definition 13.2.1. We say that (σ,μ) is *uniformly Bayesian relative to* $\precsim$ if there exists a full-support common prior $\nu : D \to (0,1]$ of $\{\nu_E\}_{E\in\mathscr{E}_\mu^+}$ that satisfies the following properties.

$UB1$. If $a \in A(h)$ and $ha \in D$, then
(i) $\nu(ha) \leq \nu(h)$ and, (ii) if $\sigma(a) > 0$ then $\nu(ha) = \nu(h)\times\sigma(a)$.

$UB2$. If $a \in A(h)$, h and h' belong to the same information set and $ha, h'a \in D$ then $\frac{\nu(h)}{\nu(h')} = \frac{\nu(ha)}{\nu(h'a)}$.

We call such a function ν a *uniform full-support common prior* of $\{\nu_E\}_{E\in\mathscr{E}_\mu^+}$.

Property $UB1$ requires that the common prior ν be consistent with the strategy profile σ, in the sense that if $\sigma(a) > 0$ then $\nu(ha) = \nu(h)\times\sigma(a)$ (thus extending Property $B2$ of Definition 13.2.1 from D_μ^+ to D).
Property $UB2$ requires that the relative probability, according to the common prior ν, of any two histories that belong to the same information set remain unchanged by the addition of the same action.

We can obtain a strengthening of the notion of perfect Bayesian equilibrium (Definition 13.3.1) by (1) adding property (IND_1) and (2) strengthening Bayes consistency (Definition 13.2.1) to uniform Bayesian consistency (Definition 13.4.1).

Definition 13.4.2 Given an extensive-form game, an assessment (σ,μ) is a *weakly independent perfect Bayesian equilibrium* if it is sequentially rational, it is rationalized by a plausibility order that satisfies (IND_1) and is uniformly Bayesian relative to that plausibility order.

As an example of a weakly independent PBE consider the game of Figure 13.7 and the assessment (σ,μ) where $\sigma = (c,d,g,\ell)$ (highlighted by double edges) and $\mu(b) = \mu(ae) = \mu(bf) = 1$ (thus $\mu(a) = \mu(af) = \mu(be) = 0$; the decision histories x such that $\mu(x) > 0$ are shown as black nodes and the decision histories x such that $\mu(x) = 0$ are shown as gray nodes).
This assessment is sequentially rational and is rationalized by the plausibility order (13.2).
It is straightforward to check that plausibility order (13.2) satisfies (IND_1).[6]

To see that (σ,μ) is uniformly Bayesian relative to plausibility order (13.2), note that $D_\mu^+ = \{\emptyset, b, ae, bf\}$ and thus the only equivalence classes that have a non-empty intersection with D_μ^+ are $E_1 = \{\emptyset, c\}$, $E_2 = \{b, bd\}$, $E_3 = \{ae, aeg\}$ and $E_4 = \{bf, bf\ell\}$.
Letting $\nu_{E_1}(\emptyset) = 1$, $\nu_{E_2}(b) = 1$, $\nu_{E_3}(ae) = 1$ and $\nu_{E_4}(bf) = 1$, this collection of probability distributions satisfies the Properties of Definition 13.2.1. Let ν be the uniform distribution over the set of decision histories $D = \{\emptyset, a, b, ae, af, be, bf\}$ (thus $\nu(h) = \frac{1}{7}$ for every $h \in D$). Then ν is a full support common prior of the collection $\{\nu_{E_i}\}_{i\in\{1,2,3,4\}}$ and satisfies Properties $UB1$ and $UB2$ of Definition 13.4.1.

[6] We have that (1) $b \succ a$, $bd \succ ad$, $be \succ ae$ and $bf \succ af$, (2) $ae \succ af$, $aeg \succ afg$ and $aek \succ afk$, (3) $bf \succ be$, $bf\ell \succ be\ell$ and $bfm \succ bem$.

$$\begin{pmatrix} \text{most plausible} & \emptyset, c \\ & b, bd \\ & a, ad \\ & bf, bf\ell \\ & be, be\ell \\ & ae, aeg \\ & af, afg \\ & bfm \\ & bem \\ & aek \\ \text{least plausible} & afk \end{pmatrix} \tag{13.2}$$

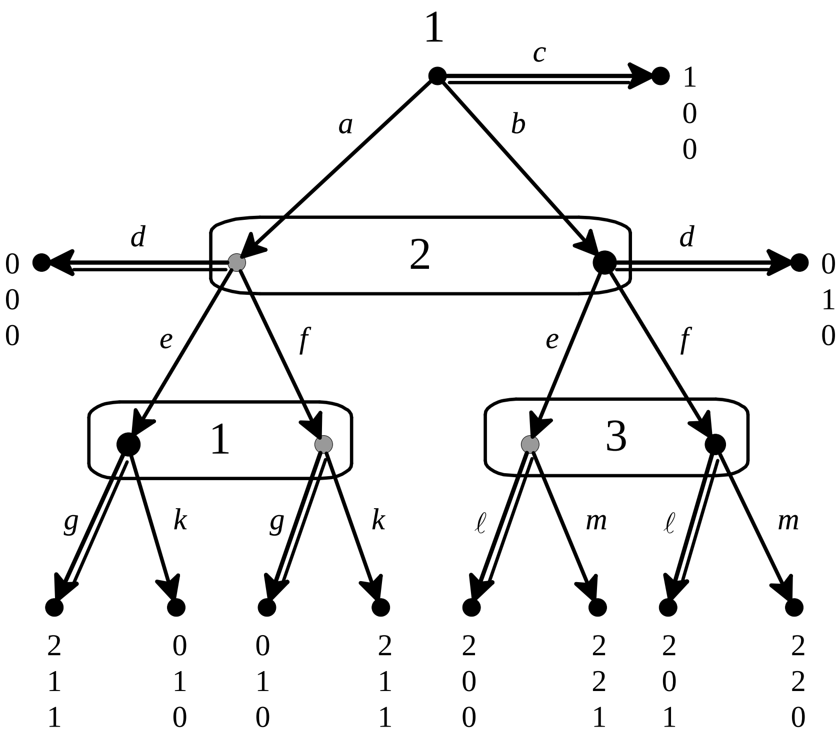


Figure 13.7: The assessment $\sigma = (c,d,g,\ell), \mu(b) = \mu(ae) = \mu(bf) = 1$ is a weakly independent PBE.

Note, however, that (σ,μ) is not a sequential equilibrium. The reader is asked to prove this in Exercise 13.9.

13.4.2 Strong independence

A second independence condition (besides (IND_1)) is Property (IND_2) below, which says that if action a is implicitly judged to be at least as plausible as action b, conditional on history h (that is, $ha \succsim hb$), then the same judgment must be made conditional on any other history that belongs to the same information set as h: if $h' \in I(h)$ and $a,b \in A(h)$, then

$$ha \succsim hb \text{ if and only if } h'a \succsim h'b. \tag{IND_2}$$

R The two properties (IND_1) and (IND_2) are independent of each other, in the sense that there are plausibility orders that satisfy one of the two properties but not the other (see Exercises 13.7 and 13.8).

Adding Property (IND_2) to the properties given in Definition 13.4.2 we obtain a refinement of the notion of weakly independent perfect Bayesian equilibrium.

Definition 13.4.3 Given an extensive-form game, an assessment (σ,μ) is a *strongly independent perfect Bayesian equilibrium* if it is sequentially rational, it is rationalized by a plausibility order that satisfies Properties (IND_1) and (IND_2), and is uniformly Bayesian relative to that plausibility order.

The notion of strongly independent PBE is a refinement of the notion of weakly independent PBE. To see this, consider again the game of Figure 13.7 and the assessment (σ,μ) where $\sigma = (c,d,g,\ell)$ and $\mu(b) = \mu(ae) = \mu(bf) = 1$ (thus $\mu(a) = \mu(af) = \mu(be) = 0$). It was shown in the previous section that (σ,μ) is a weakly independent PBE; however, it is not a strongly independent PBE because any plausibility order $\succsim$ that rationalizes (σ,μ) must violate (IND_2). In fact, since $\mu(ae) > 0$ and $\mu(af) = 0$ it follows from Property P2 of Definition 13.1.3 that

$$ae \succ af. \tag{13.3}$$

Similarly, since $\mu(bf) > 0$ and $\mu(be) = 0$ it follows from Property P2 of Definition 13.1.3 that

$$bf \succ be. \tag{13.4}$$

Since a and b belong to the same information set, (13.3) and (13.4) constitute a violation of (IND_2).

The following result states that the notions of weakly/strongly independent PBE identify two (nested) solution concepts that lie strictly in the gap between PBE and sequential equilibrium.

Theorem 13.4.1 Consider a finite extensive-form game and an assessment (σ,μ).

(A) If (σ,μ) is a sequential equilibrium then it is a strongly independent perfect Bayesian equilibrium.

(B) There are games where there is a strongly independent perfect Bayesian equilibrium which is not a sequential equilibrium.

Part **(A)** of Theorem 13.4.1 is proved in Section 13.7.
For an example of a strongly independent PBE which is not a sequential equilibrium, consider the game of Figure 13.8 and the assessment (σ,μ) with

$$\sigma = (M,\ell,a,c,e) \quad \text{(highlighted by double edges) and}$$

$$\mu = \left(\begin{array}{ccc|cc|cc|cc} L & M & R & L\ell & Mr & Lm & Rr & Mm & R\ell \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

(the decision histories h such that $\mu(h) = 1$ are denoted by large black dots and the decision histories h such that $\mu(h) = 0$ are denoted by small grey dots). In Exercise 13.13 the reader is asked to prove that (σ,μ) is a strongly independent perfect Bayesian equilibrium but not a sequential equilibrium.

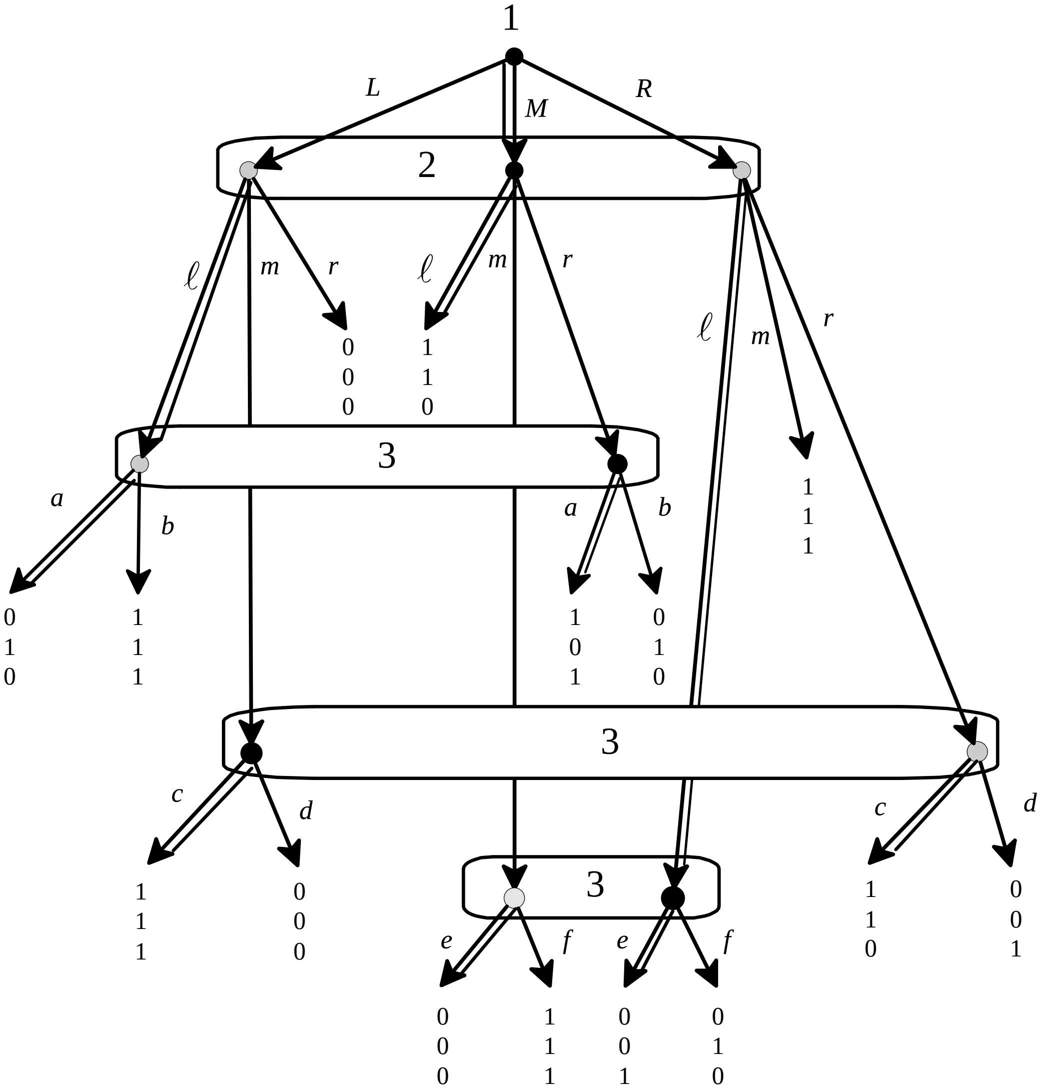


Figure 13.8: A game with a strongly independent PBE which is not a sequential equilibrium.

The relationship between the notions of (weakly/strongly independent) perfect Bayesian equilibrium and sequential equilibrium is illustrated in the Venn diagram of Figure 13.9.

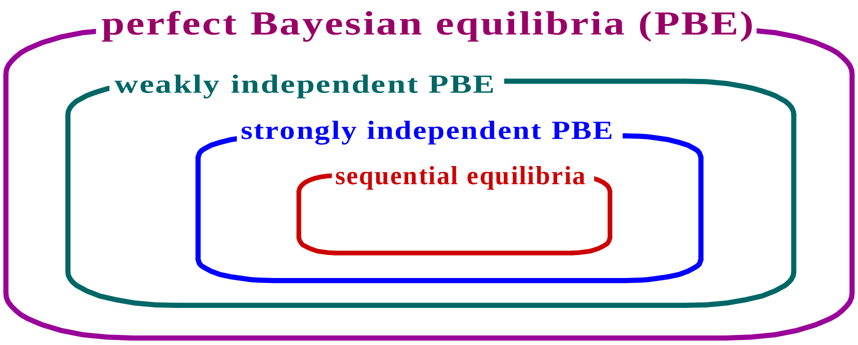


Figure 13.9: The relationship between (weakly/strongly independent) PBE and sequential equilibrium.

> Test your understanding of the concepts introduced in this section, by going through the exercises in Section 13.8.4 at the end of this chapter.

13.5 Characterization of SE in terms of PBE

Besides sequential rationality, the notion of perfect Bayesian equilibrium (Definition 13.3.1) is based on two elements:

1. the qualitative notions of plausibility order and AGM-consistency and
2. the notion of Bayesian consistency relative to the plausibility order.

By adding two further qualitative properties to the plausibility order – properties (IND_1) and (IND_2) – and by strengthening Bayesian consistency to uniform Bayesian consistency (Definition 13.4.1) we then obtained two nested refinements of the notion of PBE: weakly independent and strongly independent PBE. However, we also noted that the notion of sequential equilibrium is stronger than the notion of (weakly/strictly) independent PBE.

In this section we show that by introducing a further property of the plausibility order – which is a strengthening of both (IND_1) and (IND_2) – we can obtain a characterization of sequential equilibrium. The new requirement is that the plausibility order that rationalizes the given assessment have a "cardinal" numerical representation that can be interpreted as measuring the plausibility distance between histories in a way that is preserved by the addition of a common action.

Definition 13.5.1 Given a plausibility order $\precsim$ on a finite set of histories H, a function $F : H \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of non-negative integers) is said to be an *ordinal integer-valued representation* of $\precsim$ if, for every $h, h' \in H$,

$$F(h) \leq F(h') \text{ if and only if } h \precsim h'. \tag{13.5}$$

R

- An ordinal integer-valued representation of a plausibility order is analogous to an ordinal utility function for a preference relation: it is just a numerical representation of the order. Note that, in the case of a plausibility order, we find it more convenient to assign *lower* values to *more plausible* histories (while a utility function assigns higher values to more preferred outcomes).

- If $F : H \to \mathbb{N}$ is an integer-valued representation of a plausibility order $\precsim$ on the set of histories H, without loss of generality we can assume that $F(\emptyset) = 0$ (recall that $\emptyset$ denotes the null history, which is always one of the most plausible histories).[7]

- Since H is a finite set, an integer-valued representation of a plausibility order $\precsim$ on H always exists. A natural integer-valued representation is the following, which we shall call the *canonical integer-valued representation*.
  - Define $H_0 = \{h \in H : h \precsim x, \text{ for all } x \in H\}$ (thus H_0 is the set of most plausible histories in H), and $H_1 = \{h \in H \setminus H_0 : h \precsim x, \text{ for all } x \in H \setminus H_0\}$ (thus H_1 is the set of most plausible histories among the ones that remain after removing the set H_0 from H).
  - In general, for every integer $k \geq 1$, define $H_k = \{h \in H \setminus (H_0 \cup \cdots \cup H_{k-1}) : h \precsim x, \text{ for all } x \in H \setminus (H_0 \cup \cdots \cup H_{k-1})\}$. Since H is finite, there is an $m \in \mathbb{N}$ such that $\{H_0, \ldots, H_m\}$ is a partition of H and, for every $j, k \in \mathbb{N}$, with $j < k \leq m$, and for every $h, h' \in H$, if $h \in H_j$ and $h' \in H_k$ then $h \succ h'$.
  - Define $\hat{F} : H \to \mathbb{N}$ as follows: $\hat{F}(h) = k$ if and only if $h \in H_k$; then the function $\hat{F}$ so defined is an integer-valued representation of $\precsim$ and $\hat{F}(\emptyset) = 0$.

Instead of an ordinal representation of the plausibility order $\precsim$ one could seek a *cardinal* representation which, besides (13.5), satisfies the following property: if h and h' belong to the same information set and $a \in A(h)$, then

$$F(h') - F(h) = F(h'a) - F(ha). \tag{CM}$$

If we think of F as measuring the "plausibility distance" between histories, then we can interpret (CM) as a distance-preserving condition: the plausibility distance between two histories in the same information set is preserved by the addition of the same action.

Definition 13.5.2 A plausibility order $\precsim$ on the set of histories H is *choice measurable* if it has at least one integer-valued representation that satisfies property (CM).

[7] Let $\bar{F} : H \to \mathbb{N}$ be an integer-valued representation of a plausibility order $\precsim$ and define $F : H \to \mathbb{N}$ as follows: $F(h) = \bar{F}(h) - \bar{F}(\emptyset)$. Then F is also an integer-valued representation of $\precsim$ and $F(\emptyset) = 0$.

For example, consider the extensive form of Figure 13.10 and the assessment consisting of the pure-strategy profile $\sigma = (a,e)$ (highlighted by double edges) and the system of beliefs $\mu(b) = 1,\ \mu(c) = 0$ (the grey node in Player 2's information set represents the history which is assigned zero probability).
This assessment is rationalized by the plausibility order shown below together with two integer-valued representations: $\hat{F}$ is the canonical representation explained in the above remark (page 447), while F is an alternative representation:

	$\precsim$	$\hat{F}$	F
most plausible	$\emptyset, a$	0	0
	b, be	1	1
	bf	2	3
	c, ce	3	4
	d	4	5
least plausible	cf	5	6

While $\hat{F}$ does not satisfy Property (*CM*) (since $\hat{F}(c) - \hat{F}(b) = 3 - 1 = 2 \neq \hat{F}(cf) - \hat{F}(bf) = 5 - 2 = 3$), F does satisfy Property (*CM*). Since there is at least one integer-valued representation that satisfies Property (*CM*), by Definition 13.5.2 the above plausibility order is choice measurable.

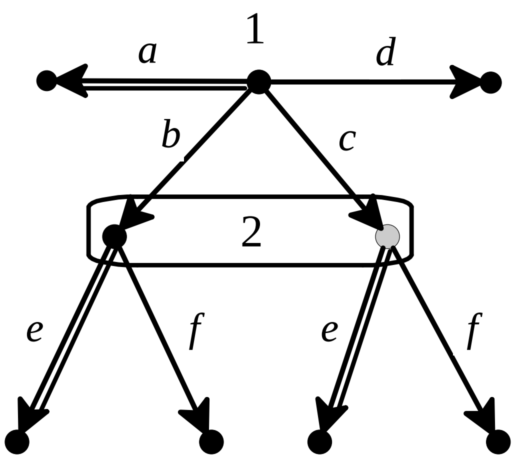


Figure 13.10: The assessment $\sigma = (a,e),\ \mu(b) = 1$ is choice measurable.

We can now state the main result of this section, namely that choice measurability and uniform Bayesian consistency are necessary and sufficient for a perfect Bayesian equilibrium to be a sequential equilibrium.

Theorem 13.5.1 — Bonanno, 2016. Consider a finite extensive-form game and an assessment (σ,μ). The following are equivalent:

(A) (σ,μ) is a sequential equilibrium,

(B) (σ,μ) is a perfect Bayesian equilibrium that is rationalized by a choice-measurable plausibility order and is uniformly Bayesian relative to it.

Theorem 13.5.1 provides a characterization (or understanding) of sequential equilibrium which is free of the questionable requirement of taking a limit of sequences of completely mixed strategies and associated systems of beliefs (see the remarks in Chapter 12, Section 12.3).

> Test your understanding of the concepts introduced in this section, by going through the exercises in Section 13.8.5 at the end of this chapter.

13.6 History-based definition of extensive-form game

If A is a set, we denote by A^* the set of finite sequences in A. If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $1 \leq j \leq k$, the sequence $h' = \langle a_1, ..., a_j \rangle$ is called a *prefix* of h (a *proper prefix* of h if $j < k$). If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $a \in A$, we denote the sequence $\langle a_1, ..., a_k, a \rangle \in A^*$ by ha.

A *finite extensive form* is a tuple $\langle A, H, I, \iota, \{\approx_i\}_{i \in I} \rangle$ whose elements are:

- A finite set of actions A.
- A finite set of histories $H \subseteq A^*$ which is closed under prefixes (that is, if $h \in H$ and $h' \in A^*$ is a prefix of h, then $h' \in H$). The null (or empty) history $\langle \rangle$, denoted by $\emptyset$, is an element of H and is a prefix of every history (the null history $\emptyset$ represents the root of the tree).
  A history $h \in H$ such that, for every $a \in A$, $ha \notin H$, is called a *terminal history*. The set of terminal histories is denoted by Z.
  $D = H \backslash Z$ denotes the set of non-terminal or *decision* histories. For every decision history $h \in D$, we denote by $A(h)$ the set of actions available at h, that is, $A(h) = \{a \in A : ha \in H\}$.
- A finite set $I = \{1, \ldots, n\}$ of players. In some cases there is also an additional, fictitious, player called *chance*.
- A function $\iota : D \to I \cup \{chance\}$ that assigns a player to each decision history. Thus $\iota(h)$ is the player who moves at history h.
  A game is said to be *without chance moves* if $\iota(h) \in I$ for every $h \in D$.
  For every $i \in I \cup \{chance\}$, let $D_i = \iota^{-1}(i)$ be the histories assigned to Player i. Thus $\{D_{chance}, D_1, \ldots, D_n\}$ is a partition of D. If history h is assigned to chance, then a probability distribution over $A(h)$ is given that assigns positive probability to every $a \in A(h)$.
- For every player $i \in I$, $\approx_i$ is an equivalence relation on D_i. The interpretation of $h \approx_i h'$ is that, when choosing an action at history $h \in D_i$, Player i does not know whether she is moving at h or at h'.
  The equivalence class of $h \in D_i$ is denoted by $I_i(h)$ and is called an *information set of Player i*; thus $I_i(h) = \{h' \in D_i : h \approx_i h'\}$. The following restriction applies: if $h' \in I_i(h)$ then $A(h') = A(h)$, that is, the set of actions available to a player is the same at any two histories that belong to the same information set of that player.
- The following property, known as *perfect recall*, is assumed: for every player $i \in I$, if $h_1, h_2 \in D_i$, $a \in A(h_1)$ and $h_1 a$ is a prefix of h_2 then for every $h' \in I_i(h_2)$ there exists an $h \in I_i(h_1)$ such that ha is a prefix of h'. Intuitively, perfect recall requires a player to remember what she knew in the past and what actions she took previously.

Given an extensive form, one obtains an *extensive game* by adding, for every Player $i \in I$, a *utility* (or *payoff*) *function* $U_i : Z \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers; recall that Z is the set of terminal histories).

Figure 13.11 shows an extensive form without chance moves where[8]

$$I = \{1,2,3,4\}, \quad A = \{a,b,s,c,d,e,f,g,h,m,n\},$$

$$Z = \{s, ace, acfg, acfh, adeg, adeh, adfm, adfn, bm, bn\},$$

$$D = \{\emptyset, a, b, ac, ad, acf, ade, adf\}, \quad H = D \cup Z,$$

$$A(\emptyset) = \{a,b,s\}, \quad A(a) = \{c,d\}, \quad A(ac) = A(ad) = \{e,f\},$$

$$A(acf) = A(ade) = \{g,h\}, \quad A(adf) = A(b) = \{m,n\},$$

$$\iota(\emptyset) = 1,\ \iota(a) = 2,\ \iota(ac) = \iota(ad) = 3,\ \iota(acf) = \iota(ade) = \iota(adf) = \iota(b) = 4,$$

$$\approx_1 = \{(\emptyset,\emptyset)\},\ \approx_2 = \{(a,a)\},\ \approx_3 = \{(ac,ac),(ac,ad),(ad,ac),(ad,ad)\},$$

$$\approx_4 = \{(acf,acf),(acf,ade),(ade,acf),(ade,ade),$$
$$(adf,adf),(adf,b),(b,adf),(b,b)\}.$$

The information sets containing more than one history (for example, $I_4(b) = \{adf, b\}$) are shown as rounded rectangles. The root of the tree represents the null history $\emptyset$.

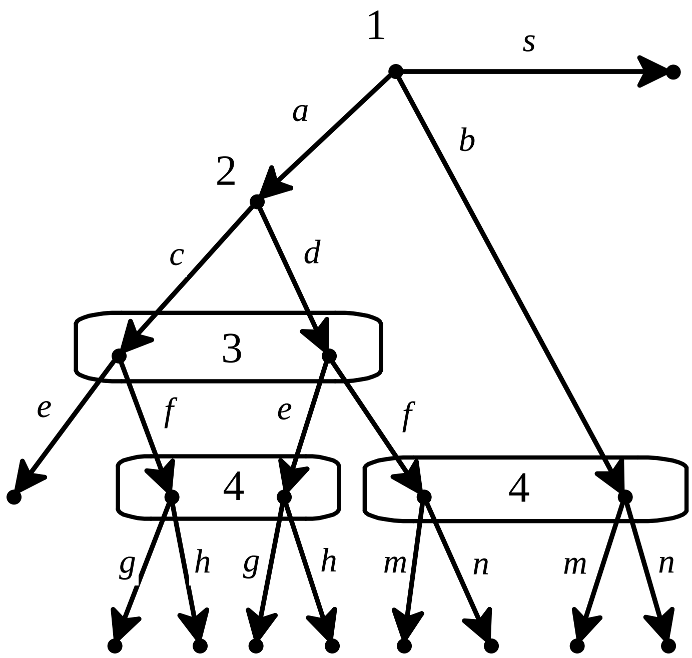


Figure 13.11: Extensive form without chance moves.

[8] In order to simplify the notation we write a instead of $\langle \emptyset, a \rangle$, ac instead of $\langle \emptyset, a, c \rangle$, etc.

If h and h' are decision histories not assigned to chance, we write $h' \in I(h)$ as a shorthand for $h' \in I_{\iota(h)}(h)$. Thus $h' \in I(h)$ means that h and h' belong to the same information set (of the player who moves at h). If h is a history assigned to chance, we use the convention that $I(h) = \{h\}$.

Given an extensive form, a *pure strategy* of player $i \in I$ is a function that associates with every information set of Player i an action at that information set, that is, a function $s_i : D_i \to A$ such that **(1)** $s_i(h) \in A(h)$ and **(2)** if $h' \in I_i(h)$ then $s_i(h') = s_i(h)$.

For example, one of the pure strategies of Player 4 in the extensive form illustrated in Figure 13.11 is $s_4(acf) = s_4(ade) = g$ and $s_4(adf) = s_4(b) = m$.

A *behavior strategy* of player i is a collection of probability distributions, one for each information set, over the actions available at that information set; that is, a function $\sigma_i : D_i \to \Delta(A)$ (where $\Delta(A)$ denotes the set of probability distributions over A) such that **(1)** $\sigma_i(h)$ is a probability distribution over $A(h)$ and **(2)** if $h' \in I_i(h)$ then $\sigma_i(h') = \sigma_i(h)$.

- If the game does not have chance moves, we define a behavior strategy *profile* as an n-tuple $\sigma = (\sigma_1, ..., \sigma_n)$ where, for every $i \in I$, σ_i is a behavior strategy of Player i.
- If the game has chance moves then we use the convention that a behavior strategy profile is an $(n+1)$-tuple $\sigma = (\sigma_1, ..., \sigma_n, \sigma_{chance})$ where, if h is a history assigned to chance and $a \in A(h)$ then $\sigma_{chance}(h)(a)$ is the probability associated with a.
- When there is no risk of ambiguity (e.g. because no action is assigned to more than one information set) we shall denote by $\sigma(a)$ the probability assigned to action a by the relevant component of the strategy profile σ.[9]

Note that a pure strategy is a special case of a behavior strategy where each probability distribution is degenerate.

A behavior strategy is *completely mixed at history $h \in D$* if, for every $a \in A(h)$, $\sigma(a) > 0$.

For example, in the extensive form of Figure 13.11:

a possible behavior strategy for Player 1 is $\begin{pmatrix} a & b & s \\ 0 & 0 & 1 \end{pmatrix}$, which we will more simply denote by s (it coincides with a pure strategy of Player 1)

and a possible behavior strategy of Player 2 is $\begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, which is a completely mixed strategy.

[9] If $h \in D_i$ and σ_i is the i^{th} component of σ, then $\sigma_i(h)$ is a probability distribution over $A(h)$ and if $a \in A(h)$ then $\sigma_i(h)(a)$ is the probability assigned to action a by $\sigma_i(h)$. Thus we denote $\sigma_i(h)(a)$ more simply by $\sigma(a)$.

13.7 Proofs

In this section we prove Theorem 13.4.1. To simplify the notation, we will assume that no action is available at more than one information set, that is, that if $a \in A(h) \cap A(h')$ then $h' \in I(h)$ (this is without loss of generality, because we can always rename some of the actions). We start with a preliminary result.

Lemma 13.1 Let $\precsim$ be a plausibility order over the set H of histories of an extensive form and let $F : H \to \mathbb{N}$ be an integer-valued representation of $\precsim$ (that is, for all $h, h' \in H$, $F(h) \leq F(h')$ if and only if $h \precsim h'$). Then the following are equivalent:

(A) F satisfies Property (CM).

(B) F satisfies the following property: for all $h, h' \in H$ and $a, b \in A(h)$, if $h' \in I(h)$ then

$$F(hb) - F(ha) = F(h'b) - F(h'a). \tag{CM'}$$

Proof. Let $\precsim$ be a plausibility order on the set of histories H and let $F : H \to \mathbb{N}$ be an integer-valued representation of $\precsim$ that satisfies Property (CM).
Without loss of generality (see Remark on page 447), we can assume that $F(\emptyset) = 0$.
For every decision history h and action $a \in A(h)$, define $\lambda(a) = F(ha) - F(h)$. The function $\lambda : A \to \mathbb{N}$ is well defined, since, by assumption, no action is available at more than one information set and, by (CM), if $h' \in I(h)$ then $F(h'a) - F(h') = F(ha) - F(h)$.
Then, for every history $h = \langle a_1, a_2, \ldots, a_m \rangle$, $F(h) = \sum_{i=1}^{m} \lambda(a_i)$. In fact,

$$\begin{aligned}
&\lambda(a_1) + \lambda(a_2) + \cdots + \lambda(a_m) \\
&= (F(a_1) - F(\emptyset)) + (F(a_1a_2) - F(a_1)) + \cdots + (F(a_1a_2 \ldots a_m) - F(a_1a_2 \ldots a_{m-1})) \\
&= F(a_1a_2 \ldots a_m) = F(h)
\end{aligned}$$

(recall that $F(\emptyset) = 0$). Thus, for every $h \in D$ and $a \in A(h)$, $F(ha) = F(h) + \lambda(a)$.
Hence, $F(hb) - F(ha) = F(h) + \lambda(b) - (F(h) + \lambda(a)) = \lambda(b) - \lambda(a)$
and $F(h'b) - F(h'a) = F(h') + \lambda(b) - (F(h') + \lambda(a)) = \lambda(b) - \lambda(a)$
so that $F(hb) - F(ha) = F(h'b) - F(h'a)$.
Thus we have shown that (CM) implies (CM').

Now we show the converse, namely that (CM') implies (CM).
Let $\precsim$ be a plausibility order on the set of histories H and let $F : H \to \mathbb{N}$ be an integer-valued representation of $\precsim$ that satisfies (CM').
Select arbitrary $h' \in I(h)$ and $a \in A(h)$.
Let $b \in A(h)$ be a plausibility-preserving action at h (there must be at least one such action: see Definition 13.1.2); then, $h \sim hb$ and $h' \sim h'b$.
Hence, since F is a representation of $\precsim$, $F(hb) = F(h)$ and $F(h'b) = F(h')$ so that $F(h') - F(h) = F(h'b) - F(hb)$.
By (CM'), $F(h'b) - F(hb) = F(h'a) - F(ha)$.
From the last two equalities it follows that $F(h') - F(h) = F(h'a) - F(ha)$,
that is, (CM) holds. ■

Proof of Theorem 13.4.1 Consider a finite extensive-form game and an assessment (σ, μ). Suppose that (σ, μ) is a sequential equilibrium. We want to show that (σ, μ) is a strongly independent PBE, that is, that (σ, μ) is sequentially rational, is rationalized by a plausibility order that satisfies properties (IND_1) and (IND_2) and is uniformly Bayesian relative to that plausibility order.

By Theorem 13.5.1 (σ, μ) is sequentially rational, is rationalized by a plausibility order that satisfies property (CM) and is uniformly Bayesian relative to that plausibility order.

Thus it is sufficient to prove that property (CM) implies properties (IND_1) and (IND_2).

- Proof that CM implies IND_1.
  Let $\precsim$ be a plausibility order and F an integer valued representation of $\precsim$ that satisfies property (CM).
  Let histories h and h' belong to the same information set $\left(h' \in I(h)\right)$
  and let a be an action available at h $\left(a \in A(h)\right)$.
  We need to show that

  $$h \precsim h' \quad \text{if and only if} \quad ha \precsim h'a.$$

  Suppose that $h \precsim h'$; then (by Definition 13.5.1) $F(h) \leq F(h')$, that is,

  $$F(h) - F(h') \leq 0. \tag{13.6}$$

  By (CM) it follows from (13.6) that $F(ha) - F(h'a) \leq 0$, that is, $F(ha) \leq F(h'a)$ and thus, by Definition 13.5.1, $ha \precsim h'a$.

  Conversely, if $ha \precsim h'a$ then $F(ha) \leq F(h'a)$ and thus, by (CM), $F(h) \leq F(h')$, which implies that $h \precsim h'$.

- Proof that CM implies IND_2.
  Let $h' \in I(h)$ and $a, b \in A(h)$. We need to show that

  $$ha \precsim hb \;\text{ if and only if }\; h'a \precsim h'b.$$

  By Lemma 13.1, $F(hb) - F(ha) = F(h'b) - F(h'a)$.
  If $F(hb) \leq F(ha)$ then $F(h'b) \leq F(h'a)$ and thus (by Definition 13.5.1) $hb \precsim ha$ and $h'b \precsim h'a$;
  if $F(ha) \leq F(hb)$ then $F(h'a) \leq F(h'b)$ and thus (by Definition 13.5.1) $ha \precsim hb$ and $h'a \precsim h'b$. ■

13.8 Exercises

13.8.1 Exercises for Section 13.1: Belief revision and AGM consistency

The answers to the following exercises are given in Section 13.9

Exercise 13.1

Consider the game shown in Figure 13.12. Determine if there is a plausibility order that rationalizes the following assessment (Definition 13.1.3):

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & s & c & d & e & f & g & h \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{array} \right) \mu = \left(\begin{array}{cc|cccc} ac & ad & acf & ade & adf & b \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 \end{array} \right)$$

■

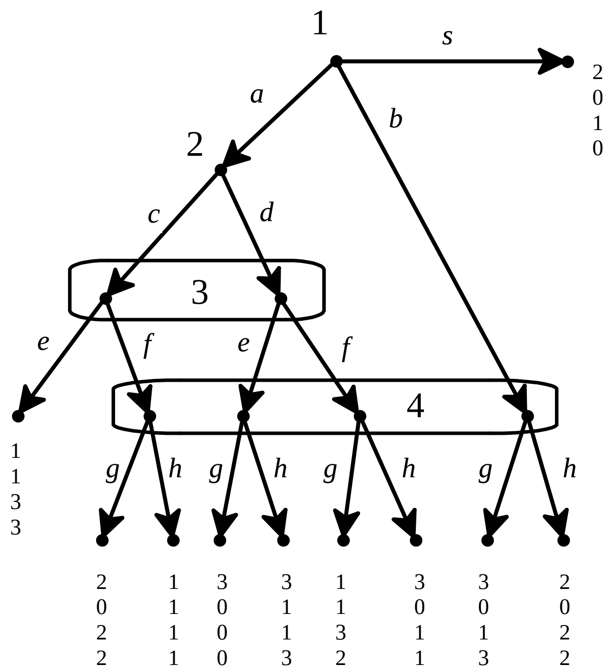


Figure 13.12: The game for Exercise 13.1.

Exercise 13.2

Consider again the game of Exercise 13.1 (Figure 13.12). Find all the assessments that are rationalized by the following plausibility order:

$$\left(\begin{array}{ll} \text{most plausible} & \emptyset, s \\ & a, ac, ad, ace, ade, adeg, b, bg \\ & acf, adf, acfg, adfg \\ & adeh, bh \\ \text{least plausible} & acfh, adfh \end{array} \right)$$

■

13.8.2 Exercises for Section 13.2: Bayesian consistency

The answers to the following exercises are given in Section 13.9.

Exercise 13.3

Consider the game of Figure 13.13 (which reproduces Figure 13.4) and the assessment

$$\sigma = \left(\begin{array}{ccc|cc|cc} a & b & c & d & e & f & g \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}\right), \quad \mu = \left(\begin{array}{ccc|ccc} ad & ae & b & a & bf & bg \\ \frac{1}{4} & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{array}\right)$$

which is rationalized by the plausibility order

$$\left(\begin{array}{ll} \text{most plausible} & \emptyset, c \\ & a, ad, adf, adg, b, bf, bg, bfd, bgd \\ \text{least plausible} & ae, aef, aeg, bfe, bge \end{array}\right).$$

Let $F = \{a, ad, adf, adg, b, bf, bg, bfd, bgd\}$ be the middle equivalence class of the plausibility order. It was shown in Section 13.2 that the probability distribution

$$\nu_F = \left(\begin{array}{ccccc} a & ad & b & bf & bg \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & \frac{2}{8} \end{array}\right)$$

satisfies the properties of Definition 13.2.1.

Show that there is no other probability distribution on F that satisfies those properties. ■

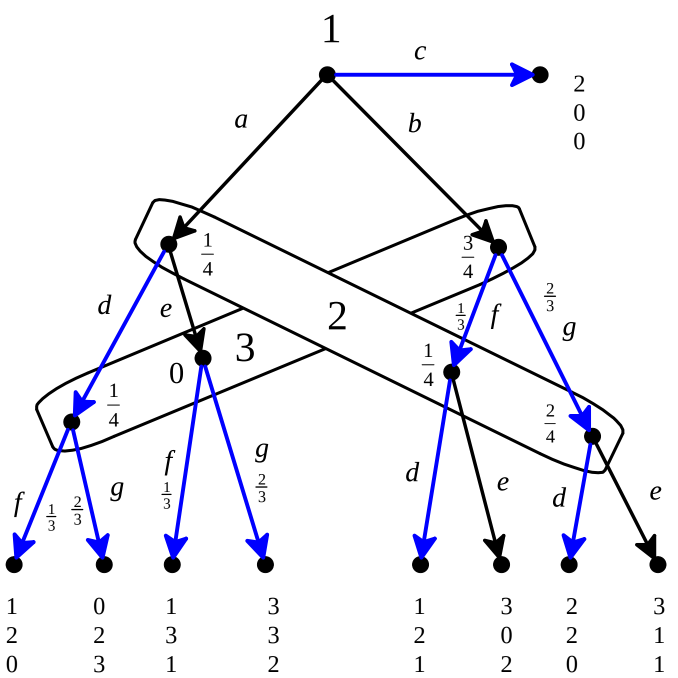


Figure 13.13: The game for Exercise 13.3.

Exercise 13.4

Consider the extensive form shown in Figure 13.14 and the assessment

$$\sigma = \left(f, A, \left(\begin{array}{cc|cc} L & R & \ell & r \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{5} & \frac{4}{5} \end{array} \right), D \right), \qquad \mu = \left(\begin{array}{ccc|cc} a & b & c & d & e \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{3}{4} & \frac{1}{4} \end{array} \right).$$

(a) Verify that the given assessment is rationalized by the following plausibility order:

$$\left(\begin{array}{ll} \text{most plausible} & \emptyset, f, fA \\ & b, c, d, e, bL, bR, cL, cR, d\ell, dr, e\ell, er, erD \\ \text{least plausible} & a, aL, aR, fB, erC \end{array} \right)$$

(b) Let $E = \{\emptyset, f, fA\}$ be the top equivalence class of the plausibility order.
Show that there is a unique probability distribution ν_E on E that satisfies the properties of Definition 13.2.1.

(c) Let $F = \{b, c, d, e, bL, bR, cL, cR, d\ell, dr, e\ell, er, erD\}$ be the middle equivalence class of the plausibility order.
Show that both of the following probability distributions satisfy the properties of Definition 13.2.1:

$$\nu_F = \left(\begin{array}{ccccc} b & c & d & e & er \\ \frac{20}{132} & \frac{40}{132} & \frac{45}{132} & \frac{15}{132} & \frac{12}{132} \end{array} \right) \qquad \hat{\nu}_F = \left(\begin{array}{ccccc} b & c & d & e & er \\ \frac{5}{87} & \frac{10}{87} & \frac{45}{87} & \frac{15}{87} & \frac{12}{87} \end{array} \right)$$

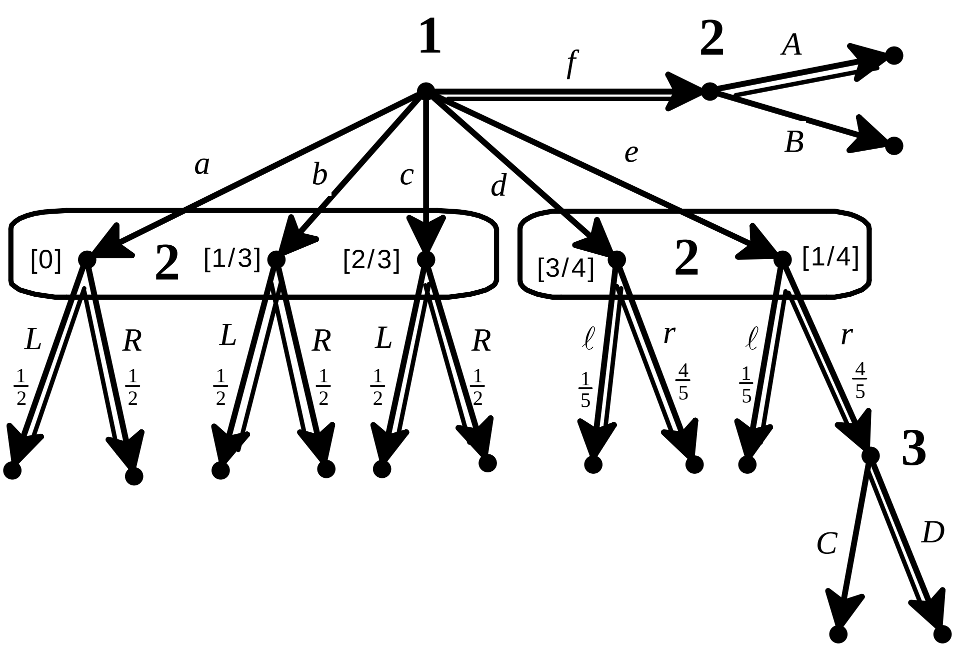


Figure 13.14: The game for Exercise 13.4.

13.8.3 Exercises for Section 13.3: Perfect Bayesian equilibrium

The answers to the following exercises are given in Section 13.9

Exercise 13.5

Consider the extensive form shown in Figure 13.15.
Find two perfect Bayesian equilibria (σ, μ) and (σ', μ') where both σ and σ' are pure-strategy profiles and $\sigma \neq \sigma'$. ■

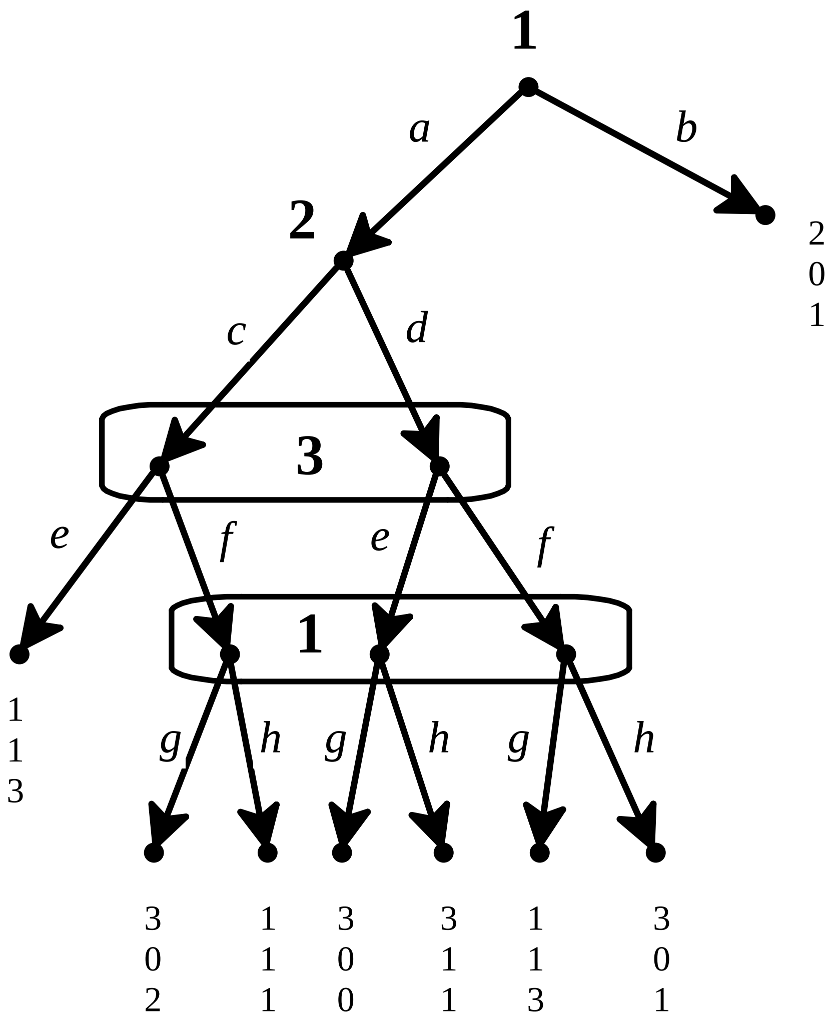


Figure 13.15: The game for Exercise 13.5.

Exercise 13.6

Consider the extensive form shown in Figure 13.16.
Prove that the following assessment is a perfect Bayesian equilibrium:

$$\sigma = \left(s, \left| \begin{array}{cc} c & d \\ \frac{1}{2} & \frac{1}{2} \end{array} \right| \begin{array}{cc} e & f \\ \frac{1}{3} & \frac{2}{3} \end{array} \left| \begin{array}{cc} g & h \\ \frac{1}{2} & \frac{1}{2} \end{array} \right. \right), \qquad \mu = \left(\begin{array}{cc} ac & ad \\ \frac{1}{2} & \frac{1}{2} \end{array} \left| \begin{array}{cccc} acf & ade & adf & b \\ \frac{2}{11} & \frac{1}{11} & \frac{2}{11} & \frac{6}{11} \end{array} \right. \right).$$

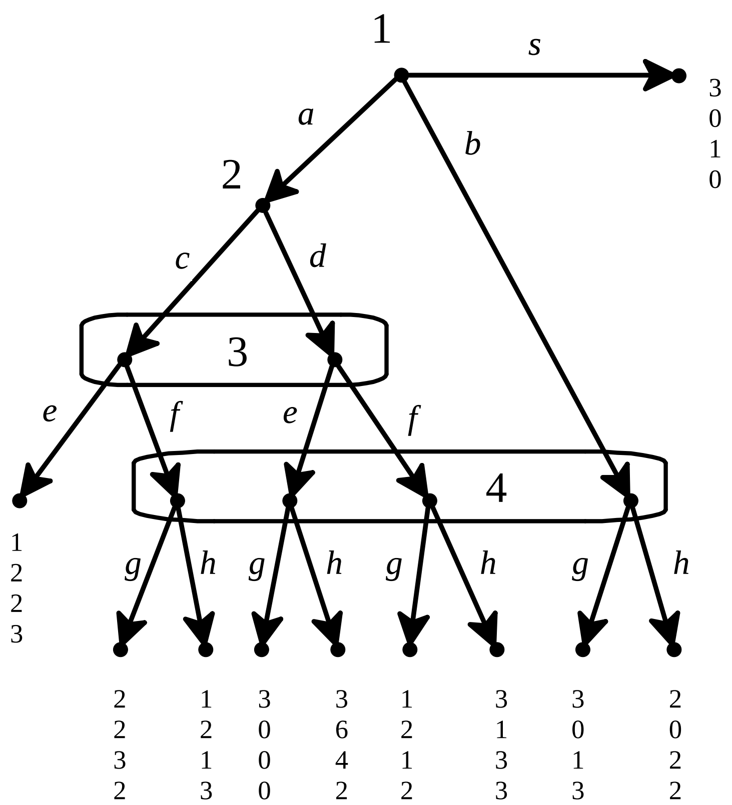


Figure 13.16: The game for Exercise 13.6.

13.8.4 Exercises for Section 13.4: Adding independence

The answers to the following exercises are given in Section 13.9

Exercise 13.7

Draw an extensive form where Player 1 moves first and Player 2 moves second without being informed of Player 1's choice.
Player 1 chooses between a and b, while Player 2's choices are c, d and e.
Find an assessment (σ, μ) which is rationalized by a plausibility order that satisfies Property (IND_1) but fails Property (IND_2).

Exercise 13.8

Find an extensive form and an assessment (σ, μ) which is rationalized by a plausibility order that violates Property (IND_1) but satisfies Property (IND_2) as well as the following property:

$$\text{if} \quad h' \in I(h), \quad a \in A(h), \quad h'a \in I(ha) \quad \text{and} \quad \{h, h', ha, h'a\} \subseteq D_\mu^+$$
$$\text{then} \quad \frac{\mu(h)}{\mu(h')} = \frac{\mu(ha)}{\mu(h'a)}.$$

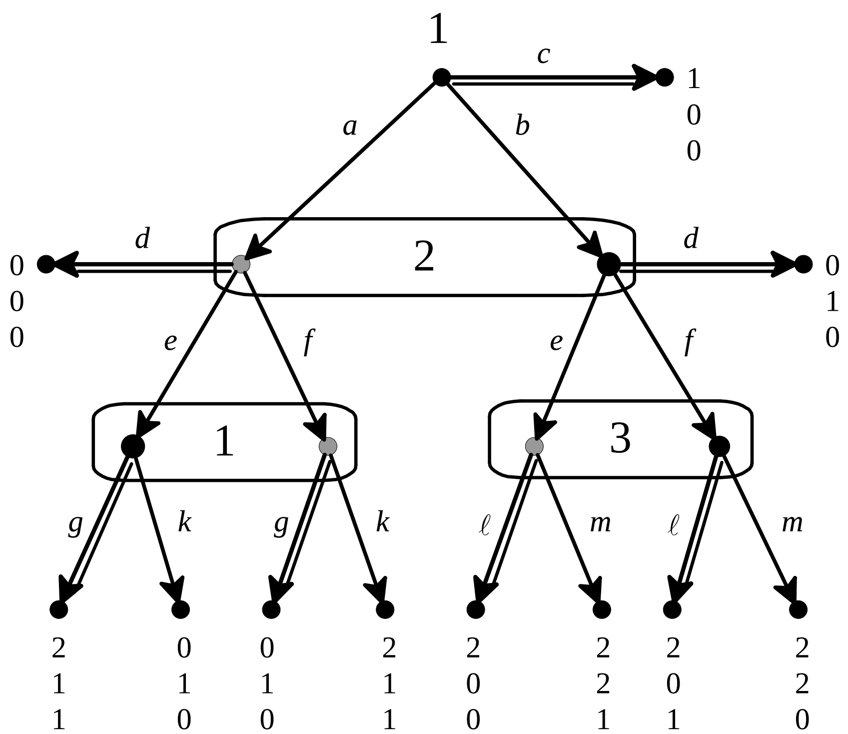
■

13.8.5 Exercises for Section 13.5: Characterization of sequential equilibrium in terms of PBE

The answers to the following exercises are given in Section 13.9

Exercise 13.9

Consider the game of Figure 13.7, reproduced below, and the assessment (σ,μ) where $\sigma = (c, d, g, \ell)$ (highlighted by double edges) and $\mu(b) = \mu(ae) = \mu(bf) = 1$ (thus $\mu(a) = \mu(af) = \mu(be) = 0$; the decision histories x such that $\mu(x) > 0$ are shown as black nodes and the decision histories x such that $\mu(x) = 0$ are shown as gray nodes). It was shown in Section 13.4.1 that (σ,μ) is a weakly independent perfect Bayesian equilibrium. Using Theorem 13.5.1 prove that (σ,μ) is not a sequential equilibrium. ■



Exercise 13.10

Consider the (partial) extensive form shown in Figure 13.17.
Using Theorem 13.5.1, prove that there is no sequential equilibrium (σ, μ) where $\sigma = (a, g, r, \ldots)$ (that is, σ assigns probability 1 to a, g and r), $\mu(c) = \mu(e) = \mu(ch) = 0$ and $\mu(h) > 0$ for every other decision history h.
[Hint: consider all the possible plausibility orders that rationalize (σ, μ).] ■

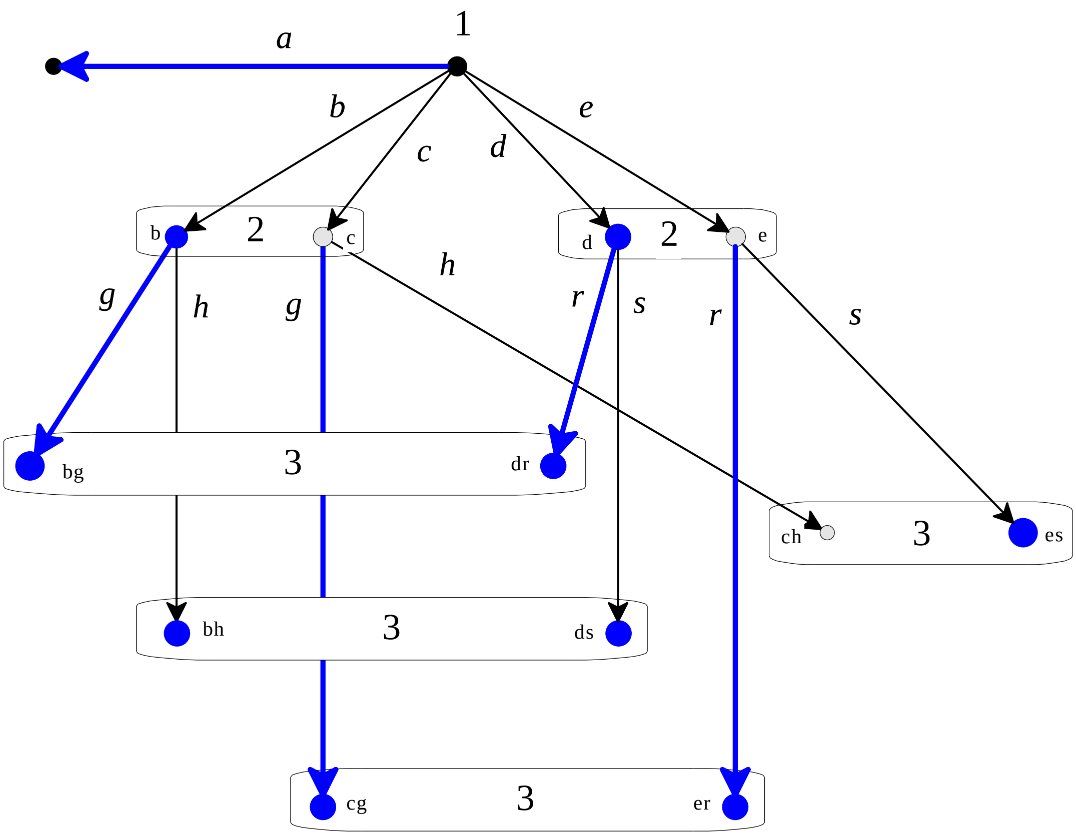


Figure 13.17: The partial extensive form for Exercise 13.10.

Exercise 13.11

Consider the game shown in Figure 13.18.
Let (σ, μ) be an assessment with $\sigma = (a, T, f, L)$ (highlighted by double edges; note that σ is a subgame-perfect equilibrium), $\mu(b) > 0$ and $\mu(c) > 0$.

(a) Prove that (σ, μ) can be rationalized by a choice-measurable plausibility order only if μ satisfies the following condition:

$$\mu(bB) > 0 \quad \text{if and only if} \quad \mu(cBf) > 0.$$

(b) Prove that if, besides from being rationalized by a choice-measurable plausibility order $\precsim$, (σ, μ) is also uniformly Bayesian relative to $\precsim$ (Definition 13.4.1), then μ satisfies the following condition:

$$\text{if} \quad \mu(bB) > 0 \quad \text{then} \quad \frac{\mu(cBf)}{\mu(bB)} = \frac{\mu(c)}{\mu(b)}.$$

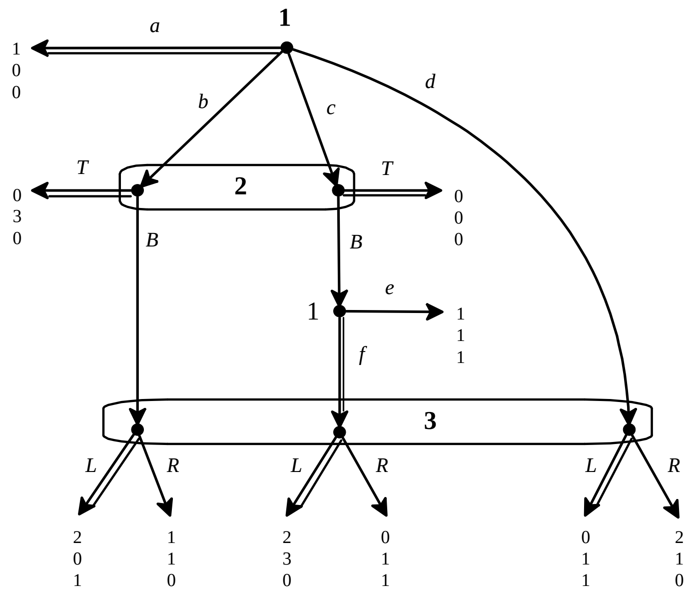


Figure 13.18: The extensive form for Exercise 13.11.

Exercise 13.12

Consider again the game of Exercise 13.11 (shown in Figure 13.18). Let (σ, μ) be an assessment with

$$\sigma = (a, T, f, L) \qquad \mu = \left(\begin{array}{cc|ccc} b & c & bB & cBf & d \\ \frac{7}{10} & \frac{3}{10} & \frac{7}{18} & \frac{3}{18} & \frac{8}{18} \end{array} \right).$$

Prove that (σ, μ) is a sequential equilibrium by using Theorem 13.5.1.

Exercise 13.13 — ⋆⋆⋆Challenging Question ⋆⋆⋆.

Consider the game shown in Figure 13.19. Let (σ,μ) be the following assessment:

$$\sigma = (M, \ell, a, c, e) \quad \text{(highlighted by double edges)}$$

$$\mu = \left(\begin{array}{ccc|cc|cc|cc} L & M & R & L\ell & Mr & Lm & Rr & Mm & R\ell \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

(the decision histories h such that $\mu(h) = 1$ are denoted by large black dots and the decision histories h such that $\mu(h) = 0$ are denoted by small grey dots).

(a) Prove that (σ,μ) is a strongly independent perfect Bayesian equilibrium (Definition 13.4.3).

(b) Prove that (σ,μ) is not a sequential equilibrium.
[Hint: prove that (σ,μ) cannot be rationalized by a choice-measurable plausibility order and then invoke Theorem 13.5.1.]

■

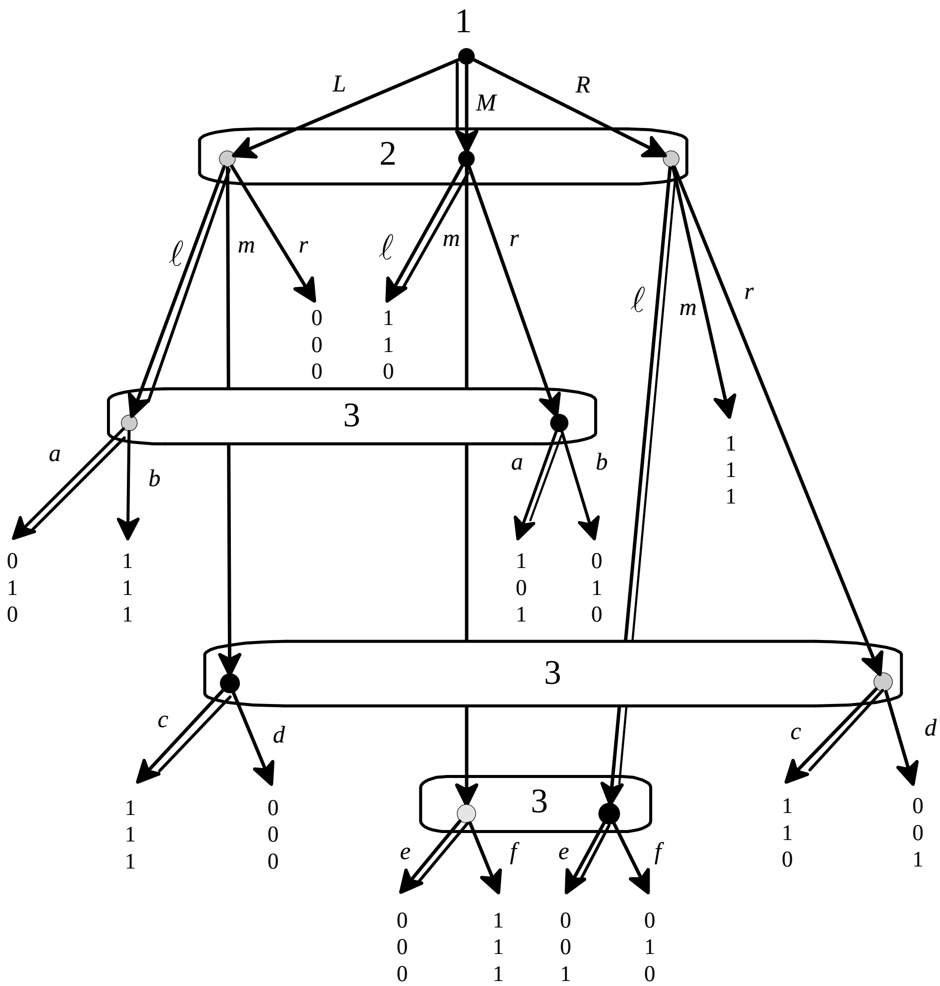


Figure 13.19: The extensive form for Exercise 13.13.

13.9 Solutions to Exercises

Solutions to Exercise 13.1

From σ we get that the plausibility preserving actions are a, s, d, e, f and g.

Thus ade is as plausible as adf (by transitivity, since each of them is as plausible as ad) and each of them is more plausible than:

- acf [since **(1)** ad is more plausible than ac, **(2)** ade and adf are as plausible as ad and **(3)** acf is as plausible as ac]
- and b [since **(1)** a is more plausible than b and **(2)** ade and adf are as plausible as ad, which in turn is as plausible as a].

Hence the most plausible histories in Player 4's information set are ade and adf.
It follows that it must be that $\mu(ade) > 0$, $\mu(adf) > 0$, $\mu(acf) = 0$ and $\mu(b) = 0$.
Hence the given assessment cannot be rationalized by a plausibility order. $\square$

Solutions to Exercise 13.2

The assessment under consideration is:

$$\begin{pmatrix} \text{most plausible} & \emptyset, s \\ & a, ac, ad, ace, ade, adeg, b, bg \\ & acf, adf, acfg, adfg \\ & adeh, bh \\ \text{least plausible} & acfh, adfh \end{pmatrix}$$

The plausibility-preserving actions are s, c, d, e and g.
Thus the strategy profile must be of the form

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & s & c & d & e & f & g & h \\ 0 & 0 & 1 & p & 1-p & 1 & 0 & 1 & 0 \end{array}\right)$$

with $0 < p < 1$.

At Player 3's information set ac and ad are equally plausible and
at Player 4's information set ade and b are equally plausible and each is more plausible than acf and adf.
Thus the system of beliefs must be of the form

$$\mu = \left(\begin{array}{cc|cccc} ac & ad & acf & ade & adf & b \\ q & 1-q & 0 & r & 0 & 1-r \end{array}\right)$$

with $0 < q < 1$ and $0 < r < 1$. $\square$

Solutions to Exercise 13.3

Since $D^+_\mu \cap F = \{a, ad, b, bf, bg\}$, Property B1 of Definition 13.2.1 requires that $\nu_F(h) > 0$ if and only if $h \in \{a, ad, b, bf, bg\}$.

Let $\nu_F(a) = p \in (0,1)$ and $\nu_F(b) = q \in (0,1)$.

Then, by Property B2 of Definition 13.2.1,
$\nu_F(ad) = \nu_F(a) \times \sigma(d) = p \times 1 = p$, $\nu_F(bf) = \nu_F(b) \times \sigma(f) = q \times \frac{1}{3}$
and $\nu_F(bg) = \nu_F(b) \times \sigma(g) = q \times \frac{2}{3}$.

Thus $\nu_F = \begin{pmatrix} a & ad & b & bf & bg \\ p & p & q & \frac{1}{3}q & \frac{2}{3}q \end{pmatrix}$

and the sum of these probabilities must be 1: $2p + 2q = 1$, that is $\boxed{p + q = \frac{1}{2}}$.

Let $I_2 = \{a, bf, bg\}$ be the information set of Player 2
and $I_3 = \{b, ad, ae\}$ the information set of Player 3.

Then $\nu_F(I_2) = \nu_F(a) + \nu_F(bf) + \nu_F(bg) = p + \frac{1}{3}q + \frac{2}{3}q = p + q = \frac{1}{2}$

and $\nu_F(I_3) = \nu_F(b) + \nu_F(ad) + \nu_F(ae) = q + p + 0 = \frac{1}{2}$.

Thus $\frac{\nu_F(a)}{\nu_F(I_2)} = \frac{p}{\frac{1}{2}} = 2p$ and, by Property B3, we need this to be equal to $\mu(a) = \frac{1}{4}$;

solving $2p = \frac{1}{4}$ we get $p = \frac{1}{8}$.

Similarly, $\frac{\nu_F(b)}{\nu_F(I_3)} = \frac{q}{\frac{1}{2}} = 2q$ and, by Property B3, we need this to be equal to $\mu(b) = \frac{3}{4}$;
solving $2q = \frac{3}{4}$ we get $q = \frac{3}{8}$

(alternatively, we could have derived $q = \frac{3}{8}$ from $p = \frac{1}{8}$ and $p + q = \frac{1}{2}$). □

Solutions to Exercise 13.4

(a) The plausibility-preserving actions are f, A, L, R, ℓ, r and D and these are precisely the actions that are assigned positive probability by σ.
Furthermore, the most plausible histories in information set $\{a,b,c\}$ are b and c and the two histories in information set $\{d,e\}$ are equally plausible.
This is consistent with the fact that $D_\mu^+ = \{\emptyset, f, b, c, d, e, er\}$.
Thus the properties of Definition 13.1.3 are satisfied.

(b) By Property B1 of Definition 13.2.1 the support of ν_E must be $D_\mu^+ \cap E = \{\emptyset, f\}$ and by Property B2 it must be that $\nu_E(f) = \nu_E(\emptyset) \times \sigma(f) = \nu_E(\emptyset) \times 1 = \nu_E(\emptyset)$.
Thus the only solution is $\nu_E(\emptyset) = \nu_E(f) = \frac{1}{2}$.

(c) Since $D_\mu^+ \cap F = \{b, c, d, e, er\}$:

- By Property B1 of Definition 13.2.1 the support of the probability distribution must coincide with the set $\{b,c,d,e,er\}$, which is indeed true for both ν_F and $\hat{\nu}_F$.
- By Property B2, the probability of er must be equal to the probability of e times $\sigma(r) = \frac{4}{5}$ and this is indeed true for both ν_F and $\hat{\nu}_F$.
- By Property B3, the conditional probability $\frac{\nu_F(b)}{\nu_F(a)+\nu_F(b)+\nu_F(c)} = \frac{\frac{20}{132}}{0+\frac{20}{132}+\frac{40}{132}} = \frac{1}{3}$

  must be equal to $\mu(b)$ and this is indeed true;
  similarly, the conditional probability $\frac{\nu_F(d)}{\nu_F(d)+\nu_F(e)} = \frac{\frac{45}{132}}{\frac{45}{132}+\frac{15}{132}} = \frac{3}{4}$ must be equal to $\mu(d)$ and this is also true.

  Similar computations show that $\frac{\hat{\nu}_F(b)}{\hat{\nu}_F(a)+\hat{\nu}_F(b)+\hat{\nu}_F(c)} = \mu(b)$

  and $\frac{\hat{\nu}_F(d)}{hat\nu_F(d)+\hat{\nu}_F(e)} = \mu(d)$. $\square$

Solutions to Exercise 13.5 The game under consideration is reproduced in Figure 13.20.

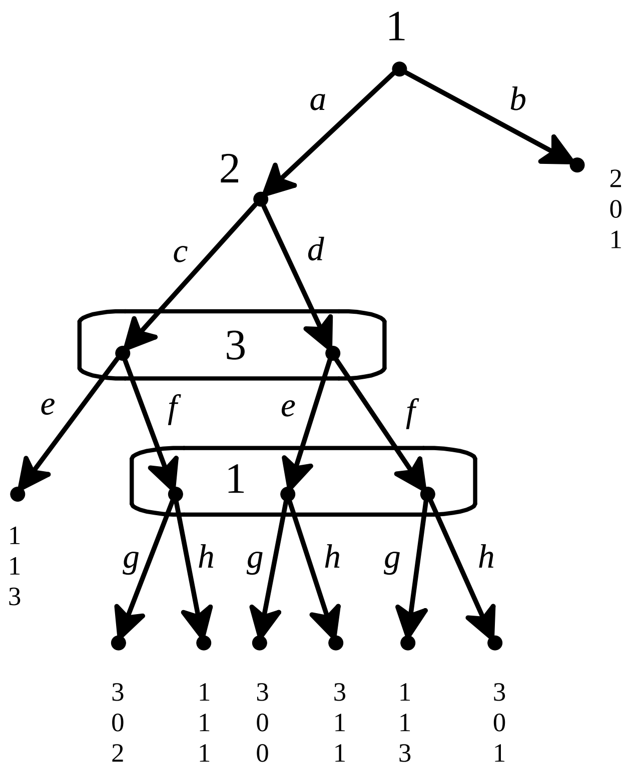


Figure 13.20: The extensive game for Exercise 13.5.

One perfect Bayesian equilibrium is

$$\sigma = (b, c, e, g), \quad \mu = \left(\begin{array}{cc|ccc} ac & ad & acf & ade & adf \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right).$$

Let us first verify sequential rationality.

For Player 1, at the root, b gives a payoff of 2, while a gives a payoff of 1; thus b is sequentially rational.

For Player 2, c gives a payoff of 1 and d a payoff of 0; thus c is sequentially rational.

For Player 3, e gives a payoff of 3, while f gives a payoff of 2; thus e is sequentially rational.

For Player 1 at his bottom information set, g gives a payoff of $\left(\frac{1}{2}\right)3 + \left(\frac{1}{2}\right)3 = 3$ and h gives a payoff of $\left(\frac{1}{2}\right)1 + \left(\frac{1}{2}\right)3 = 2$; thus g is sequentially rational.

The following plausibility order rationalizes the above assessment:

$$\begin{pmatrix} \text{most plausible} & \emptyset, b \\ & a, ac, ace \\ & acf, acfg, ad, ade, adeg \\ & adf, adfg, acfh, adeh \\ \text{least plausible} & adfh \end{pmatrix}$$

Name the equivalence classes of this order $E_1, E_2, \ldots, E_5$ (with E_1 being the top one and E_5 the bottom one).
Then the following probability distributions on the sets $E_i \cap D_\mu^+$ ($i = 1,2,3$) satisfy the properties of Definition 13.2.1 $\big($note that $E_i \cap D_\mu^+ \neq \emptyset$ if and only if $i \in \{1,2,3\}$, since $D_\mu^+ = \{\emptyset, a, ac, acf, ade\}\big)$:

$$\nu_{E_1}(\emptyset) = 1, \qquad \nu_{E_2} = \begin{pmatrix} a & ac \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad \nu_{E_3} = \begin{pmatrix} acf & ad & ade \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Another perfect Bayesian equilibrium is:

$$\sigma = (a, d, e, h), \quad \mu = \left(\begin{array}{cc|ccc} ac & ad & acf & ade & adf \\ 0 & 1 & 0 & 1 & 0 \end{array}\right).$$

Sequential rationality is easily verified. The assessment is rationalized by the following plausibility order:

$$\begin{pmatrix} \text{most plausible} & \emptyset, a, ad, ade, adeh \\ & b, ac, ace, adf, adfh \\ & acf, acfh, adeg \\ \text{least plausible} & acfg, adfg \end{pmatrix}$$

Only the top equivalence class $E_1 = \{\emptyset, a, ad, ade, adeh\}$ has a non-empty intersection with $D_\mu^+ = \{\emptyset, a, ad, ade\}$.

The following probability distribution satisfies the properties of Definition 13.2.1:
$\nu_{E_1} = \begin{pmatrix} \emptyset & a & ad & ade \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. □

Solutions to Exercise 13.6 The game under consideration is reproduced in Figure 13.21.

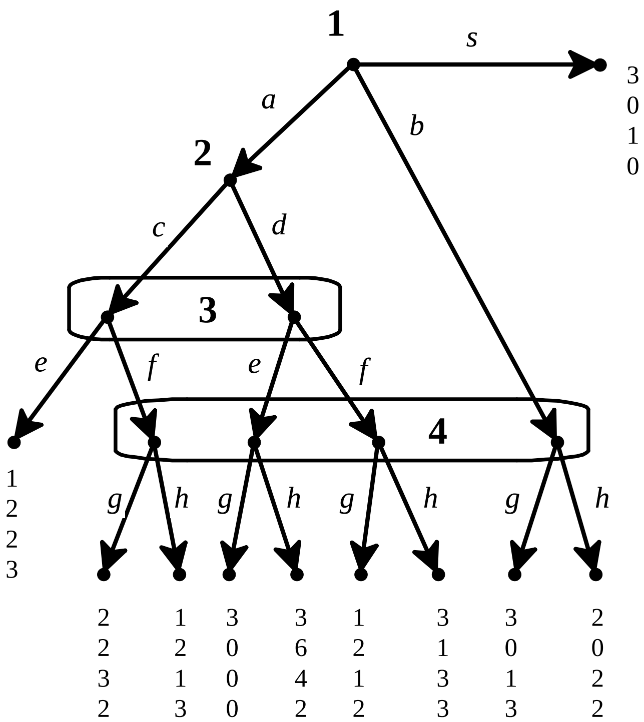


Figure 13.21: The extensive game for Exercise 13.6.

We need to show that the following is a perfect Bayesian equilibrium:

$$\sigma = \left(s, \left| \begin{array}{cc} c & d \\ \frac{1}{2} & \frac{1}{2} \end{array} \right| \begin{array}{cc} e & f \\ \frac{1}{3} & \frac{2}{3} \end{array} \left| \begin{array}{cc} g & h \\ \frac{1}{2} & \frac{1}{2} \end{array} \right.\right), \quad \mu = \left(\begin{array}{cc} ac & ad \\ \frac{1}{2} & \frac{1}{2} \end{array} \left| \begin{array}{cccc} acf & ade & adf & b \\ \frac{2}{11} & \frac{1}{11} & \frac{2}{11} & \frac{6}{11} \end{array} \right.\right).$$

First we verify sequential rationality.

For Player 1 the possible payoffs are:

$$\begin{array}{ll} \text{from } s\text{:} & 3 \\ \text{from } b\text{:} & \frac{1}{2}(3)+\frac{1}{2}(2)=2.5 \\ \text{from } a\text{:} & \frac{1}{2}\left[\frac{1}{3}(1)+\frac{2}{3}\left(\frac{1}{2}(2)+\frac{1}{2}(1)\right)\right]+\frac{1}{2}\left[\frac{1}{3}(3)+\frac{2}{3}\left(\frac{1}{2}(1)+\frac{1}{2}(3)\right)\right]=\frac{11}{6} \end{array}.$$

Thus s is sequentially rational.

For Player 2 the possible payoffs are:

$$\begin{array}{ll} \text{from } c\text{:} & \frac{1}{3}(2)+\frac{2}{3}(2)=2 \\ \text{from } d\text{:} & \frac{1}{3}\left[\frac{1}{2}(0)+\frac{1}{2}(6)\right]+\frac{2}{3}\left[\frac{1}{2}(2)+\frac{1}{2}(1)\right]=2. \end{array}$$

Thus both c and d are sequentially rational and so is any mixture of the two, in particular the mixture $\begin{pmatrix} c & d \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

For Player 3 the possible payoffs are:

$$\begin{array}{ll} \text{from } e\text{:} & \frac{1}{2}(2)+\frac{1}{2}\left[\frac{1}{2}(0)+\frac{1}{2}(4)\right]=2 \\ \text{from } f\text{:} & \frac{1}{2}\left[\frac{1}{2}(3)+\frac{1}{2}(1)\right]+\frac{1}{2}\left[\frac{1}{2}(1)+\frac{1}{2}(3)\right]=2. \end{array}$$

Thus both e and f are sequentially rational and so is any mixture of the two, in particular the mixture $\begin{pmatrix} e & f \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

For Player 4 the possible payoffs are:
from g: $\frac{2}{11}(2)+\frac{1}{11}(0)+\frac{2}{11}(2)+\frac{6}{11}(3)=\frac{26}{11}$,
from h: $\frac{2}{11}(3)+\frac{1}{11}(2)+\frac{2}{11}(3)+\frac{6}{11}(2)=\frac{26}{11}$.
Thus both g and h are rational and so is any mixture of the two, in particular the mixture $\begin{pmatrix} g & h \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

For AGM consistency, note that all of the actions, except a and b, are plausibility preserving and, furthermore, all decision histories are assigned positive probability by μ. Thus there is only one plausibility order that rationalizes the given assessment, namely the one that has only two equivalence classes: the top one being $\{\emptyset, s\}$ and the other one consisting of all the remaining histories.

For Bayesian consistency, the probability distribution for the top equivalence class is the trivial one that assigns probability 1 to the null history $\emptyset$.
Let $E = H \setminus \{\emptyset, s\}$ be the other equivalence class and note that $E \cap D_m^+ = \{a, ac, ad, acf, ade, adf, b\}$.
In order for a probability distribution ν_E to satisfy the properties of Definition 13.2.1, the support of ν_E must be the set $\{a, ac, ad, acf, ade, adf, b\}$ (Property B1).
Let $\nu_E(a)=p$; then by Property B2 it must be that
$\nu_E(ac)=\nu_E(ad)=\frac{p}{2}$ (since $\sigma(c)=\sigma(d)=\frac{1}{2}$),
$\nu_E(acf)=\nu_E(adf)=\frac{p}{3}$ (since $\sigma(c)\times\sigma(f)=\sigma(d)\times\sigma(f)=\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$) and
$\nu_E(ade)=\frac{p}{6}$ (since $\sigma(d)\times\sigma(e)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$).
Thus ν_E must be of the form

$$\begin{pmatrix} a & ac & ad & acf & ade & adf & b \\ p & \frac{p}{2} & \frac{p}{2} & \frac{p}{3} & \frac{p}{6} & \frac{p}{3} & q \end{pmatrix}$$

with the sum equal to 1, that is, $\frac{17}{6}p+q=1$. Furthermore, by Property B3, it must be that

$$\frac{\nu_E(b)}{\nu_E(acf)+\nu_E(ade)+\nu_E(adf)+\nu_E(b)}=\frac{q}{\frac{p}{3}+\frac{p}{6}+\frac{p}{3}+q}=\underbrace{\frac{6}{11}}_{=\mu(b)}.$$

The solution to the pair of equations $\frac{17}{6}p+q=1$ and $\frac{q}{\frac{p}{3}+\frac{p}{6}+\frac{p}{3}+q}=\frac{6}{11}$ is $p=q=\frac{6}{23}$, yielding

$$\nu_E = \begin{pmatrix} a & ac & ad & acf & ade & adf & b \\ \frac{6}{23} & \frac{3}{23} & \frac{3}{23} & \frac{2}{23} & \frac{1}{23} & \frac{2}{23} & \frac{6}{23} \end{pmatrix}.$$

Thus we have shown that the given assessment is AGM-consistent, Bayes-consistent and sequentially rational, hence a perfect Bayesian equilibrium. $\square$

Solutions to Exercise 13.7 The extensive form is shown in Figure 13.22.

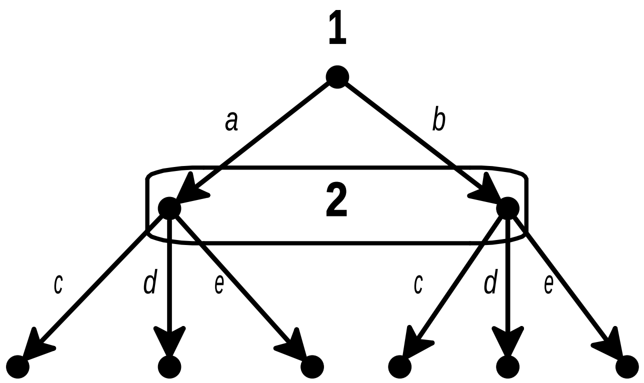


Figure 13.22: The extensive form for Exercise 13.7.

Let

$$\sigma = \left(\begin{array}{cc|ccc} a & b & c & d & e \\ 1 & 0 & 1 & 0 & 0 \end{array}\right) \qquad \mu = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

This assessment is rationalized by the following plausibility order:

$$\begin{pmatrix} \text{most plausible} & \emptyset, a, ac \\ & b, bc \\ & ad \\ & ae \\ & be \\ \text{least plausible} & bd \end{pmatrix}$$

which satisfies Property (IND_1) (since $a \succ b$ and $ac \succ bc$, $ad \succ bd$, $ae \succ be$) but fails Property (IND_2) since $b \in I(a)$ and $ad \succ ae$ (implying that – conditional on a – d is more plausible than e) but $be \succ bd$ (implying that – conditional on b – e is more plausible than d). $\square$

Solutions to Exercise 13.8 The game of Figure 13.5, reproduced in Figure 13.23, provides such an example.

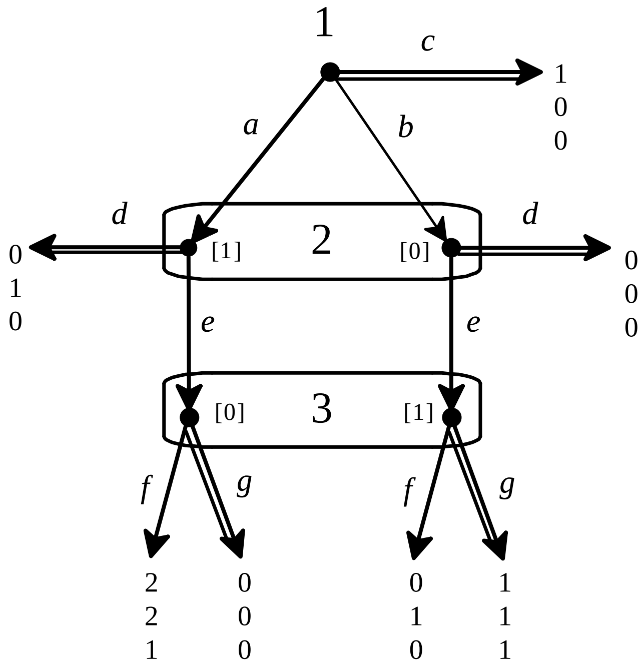


Figure 13.23: Copy of Figure 13.5.

Consider the assessment $\sigma = (c, d, g)$ (highlighted by double edges), together with the system of beliefs $\mu = \left(\begin{array}{cc|cc} a & b & ae & be \\ 1 & 0 & 0 & 1 \end{array}\right)$.

This assessment is rationalized by the plausibility order

$$\left(\begin{array}{ll} \text{most plausible} & \emptyset, c \\ & a, ad \\ & b, bd \\ & be, beg \\ & ae, aeg \\ & bef \\ \text{least plausible} & aef \end{array}\right)$$

which violates Property (IND_1) since $a \succ b$ and yet $be \succ ae$.

On the other hand, the above plausibility order satisfies Property (IND_2) since **(1)** $ad \succ ae$ and $bd \succ be$ and **(2)** $aeg \succ aef$ and $beg \succ bef$.

Furthermore, the above assessment trivially satisfies the additional property, since $a, be \in D_\mu^+$ but $ae, b \notin D_\mu^+$ (we have that $b \in I(a)$, $e \in A(a)$, $be \in I(ae)$ but it is not true that $\{a, b, ae, be\} \subseteq D_\mu^+ = \{\emptyset, a, be\}$). □

Solutions to Exercise 13.9 By Appealing to Theorem 13.5.1 it is sufficient to show that any plausibility order that rationalizes (σ, μ) cannot be choice measurable.
Let $\precsim$ be a plausibility order that rationalizes (σ, μ). Then

- since $\mu(ae) > 0$ while $\mu(af) = 0$ and ae and af belong to the same information set, by P2 of Definition 13.1.3
$$ae \succ af, \tag{13.7}$$
- since $\mu(bf) > 0$ while $\mu(be) = 0$ and bf and be belong to the same information set, by P2 of Definition 13.1.3
$$bf \succ be. \tag{13.8}$$

Let F be an integer-valued representation of $\precsim$ that satisfies (CM).
Then, since a and b belong to the same information set, it must be that
$F(b) - F(a) = F(be) - F(ae)$ and $F(b) - F(a) = F(bf) - F(af)$,
so that $F(be) - F(ae) = F(bf) - F(af)$ and thus

$$F(af) - F(ae) = F(bf) - F(be). \tag{13.9}$$

However, by definition of integer-valued representation (Definition 13.5.1), from (13.7) we get that $F(af) - F(ae) > 0$ and from (13.8) we get that $F(bf) - F(be) < 0$, contradicting (13.9). $\square$

Solutions to Exercise 13.10 The extensive form under consideration is shown in Figure 13.24.
Consider an arbitrary assessment of the form (σ, μ) where
$\sigma = (a, g, r, \dots)$, $\mu(c) = \mu(e) = \mu(ch) = 0$ and $\mu(h) > 0$ for every other decision history h.
There are plausibility orders that rationalize (σ, μ); for example, either of the following:

$$\begin{pmatrix} \text{most plausible} & \emptyset, a \\ & b, bg, d, dr \\ & bh, ds \\ & c, cg, e, er \\ & es \\ \text{least plausible} & ch \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \text{most plausible} & \emptyset, a \\ & b, bg, d, dr \\ & c, cg, e, er \\ & bh, ds \\ & es \\ \text{least plausible} & ch \end{pmatrix}.$$

First we show that any plausibility order $\precsim$ that rationalizes (σ, μ) must satisfy the following properties:

$c \sim e$ because, by P2 of Definition 13.1.3, $cg \sim er$ (since $cg \in I(er)$, $\mu(cg) > 0$ and $\mu(er) > 0$) and, by P1 of Definition 13.1.3, $c \sim cg$ and $e \sim er$ (since both g and r are plausibility preserving) and thus, by transitivity, $c \sim e$,

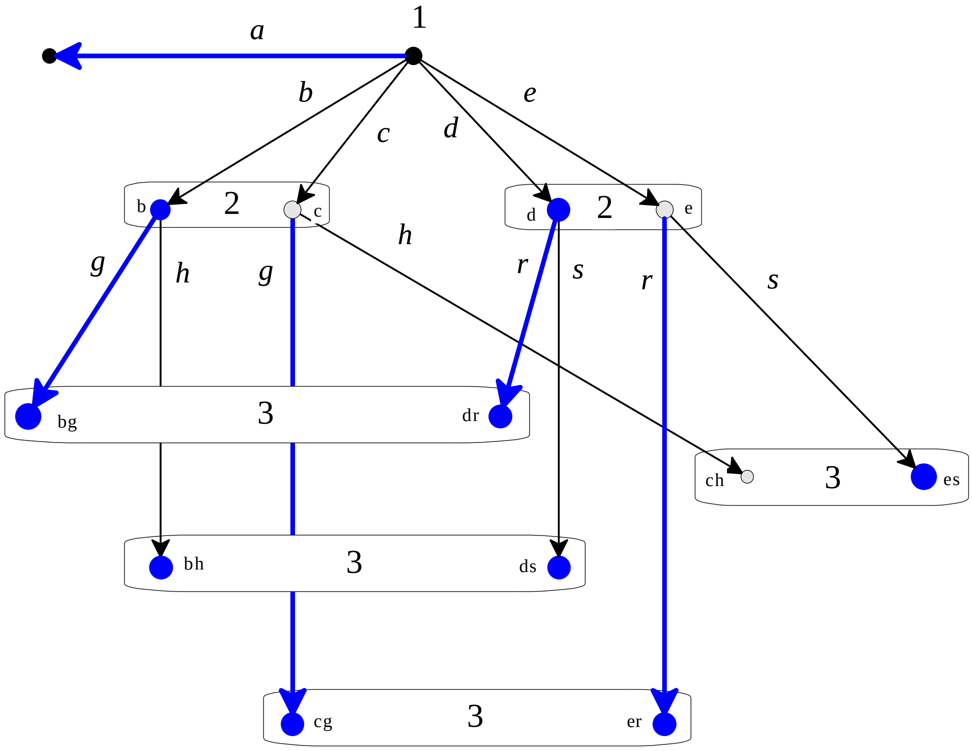


Figure 13.24: The extensive form for Exercise 13.10.

$es \succ ch$ this follows from P2 of Definition 13.1.3, since es and ch belong to the same information set and $\mu(es) > 0$, while $\mu(ch) = 0$,

$b \sim d$ because, by P2 of Definition 13.1.3, $bg \sim dr$ (since $bg \in I(dr)$, $\mu(bg) > 0$ and $\mu(dr) > 0$) and, by P1 of Definition 13.1.3, $b \sim bg$ and $d \sim dr$ (since both g and r are plausibility preserving) and thus, by transitivity, $b \sim d$,

$bh \sim ds$ by P2 of Definition 13.1.3, because $bh \in I(ds)$, $\mu(bh) > 0$ and $\mu(ds) > 0$.

Next we show that no plausibility order that rationalizes (σ, μ) is choice measurable. Select an arbitrary plausibility order that rationalizes (σ, μ) and let F be an integer valued representation of it.
Then the following must be true:

1. $F(e) - F(es) > F(c) - F(ch)$ (because $c \sim e$, implying that $F(c) = F(e)$, and $es \succ ch$, implying that $F(es) < F(ch)$),
2. $F(b) - F(bh) = F(d) - F(ds)$ (because $bh \sim ds$, implying that $F(bh) = F(ds)$, and $b \sim d$, implying that $F(b) = F(d)$).

Thus if, as required by choice measurability, $F(c) - F(ch) = F(b) - F(bh)$ then, by Points 1 and 2 above, $F(e) - F(es) > F(d) - F(ds)$, which violates choice measurability. It follows from Theorem 13.5.1 that, since any plausibility ordering that rationalizes (σ, μ) is not choice measurable, (σ, μ) cannot be a sequential equilibrium. $\square$

Solutions to Exercise 13.11 The game under consideration is shown in Figure 13.25.

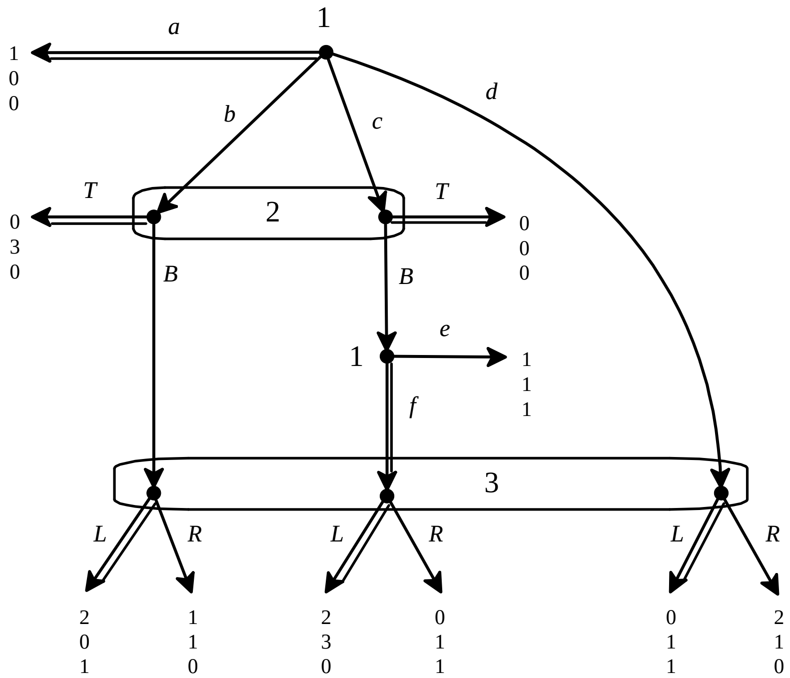


Figure 13.25: The game for Exercise 13.11.

Let (σ, μ) be an assessment with $\sigma = (a, T, f, L)$, $\mu(b) > 0$ and $\mu(c) > 0$.

(a) We have to prove that (σ, μ) can be rationalized by a choice-measurable plausibility order only if μ satisfies the following condition:

$$\mu(bB) > 0 \quad \text{if and only if} \quad \mu(cBf) > 0.$$

Let $\precsim$ be a choice measurable plausibility order that rationalizes (σ, μ) and let F be an integer-valued representation of $\precsim$ that satisfies choice measurability. Since $\mu(b) > 0$ and $\mu(c) > 0$, by P2 of Definition 13.1.3, $b \sim c$ and thus $F(b) = F(c)$; by choice measurability, $F(b) - F(c) = F(bB) - F(cB)$ and thus $F(bB) = F(cB)$, so that $bB \sim cB$.

Since $\sigma(f) > 0$, by P1 of Definition 13.1.3, $cB \sim cBf$ and therefore, by transitivity of $\precsim$, $bB \sim cBf$.

Hence if $\mu(bB) > 0$ then, by P2 of Definition 13.1.3, $bB \precsim h$ for every $h \in \{bB, cBf, d\}$ and thus (since $cBf \sim bB$) $cBf \precsim h$ for every $h \in \{bB, cBf, d\}$ so that, by P2 of Definition 13.1.3, $\mu(cBf) > 0$.

The proof that if $\mu(cBf) > 0$ then $\mu(bB) > 0$ is analogous.

(b) We have to prove that if, besides from being rationalized by a choice-measurable plausibility order $\precsim$, (σ,μ) is also uniformly Bayesian relative to $\precsim$ (Definition 13.4.1), then μ satisfies the following condition:

$$\text{if} \quad \mu(bB) > 0 \quad \text{then} \quad \frac{\mu(cBf)}{\mu(bB)} = \frac{\mu(c)}{\mu(b)}.$$

Suppose that $\mu(b) > 0$, $\mu(c) > 0$ (so that $b \sim c$) and $\mu(bB) > 0$.
Let ν be a full-support common prior that satisfies the properties of Definition 13.4.1. Then, by UB2, $\frac{\nu(c)}{\nu(b)} = \frac{\nu(cB)}{\nu(bB)}$ and, by UB1, since $\sigma(f) = 1$, $\nu(cBf) = \nu(cB) \times \sigma(f) = \nu(cB)$.

Let E be the equivalence class that contains b.
Then $E \cap D^+_\mu = \{b,c\}$.
Since $\nu_E(\cdot) = \nu(\cdot \mid E \cap D^+_\mu)$, by B3 of Definition 13.2.1, $\mu(b) = \frac{\nu(b)}{\nu(b)+\nu(c)}$ and $\mu(c) = \frac{\nu(c)}{\nu(b)+\nu(c)}$, so that $\frac{\mu(c)}{\mu(b)} = \frac{\nu(c)}{\nu(b)}$.

Let G be the equivalence class that contains bB.
Then, since – by hypothesis – $\mu(bB) > 0$, it follows from the condition proved in Part **(a)** that either $G \cap D^+_\mu = \{bB, cBf\}$ or $G \cap D^+_\mu = \{bB, cBf, d\}$.
Since $\nu_G(\cdot) = \nu(\cdot \mid G \cap D^+_\mu)$, by B3 of Definition 13.2.1,
in the former case $\mu(bB) = \frac{\nu(bB)}{\nu(bB)+\nu(cBf)}$ and $\mu(cBf) = \frac{\nu(cBf)}{\nu(bB)+\nu(cBf)}$
and in the latter case $\mu(bB) = \frac{\nu(bB)}{\nu(bB)+\nu(cBf)+\nu(d)}$ and $\mu(cBf) = \frac{\nu(cBf)}{\nu(bB)+\nu(cBf)+\nu(d)}$;
thus in both cases $\frac{\mu(cBf)}{\mu(bB)} = \frac{\nu(cBf)}{\nu(bB)}$.

Hence, since $\nu(cBf) = \nu(cB)$, $\frac{\mu(cBf)}{\mu(bB)} = \frac{\nu(cB)}{\nu(bB)}$ and, therefore, since (as shown above) $\frac{\nu(cB)}{\nu(bB)} = \frac{\nu(c)}{\nu(b)}$ and $\frac{\nu(c)}{\nu(b)} = \frac{\mu(c)}{\mu(b)}$, we have that $\frac{\mu(cBf)}{\mu(bB)} = \frac{\mu(c)}{\mu(b)}$. □

Solutions to Exercise 13.12 The game under consideration is reproduced in Figure 13.26.

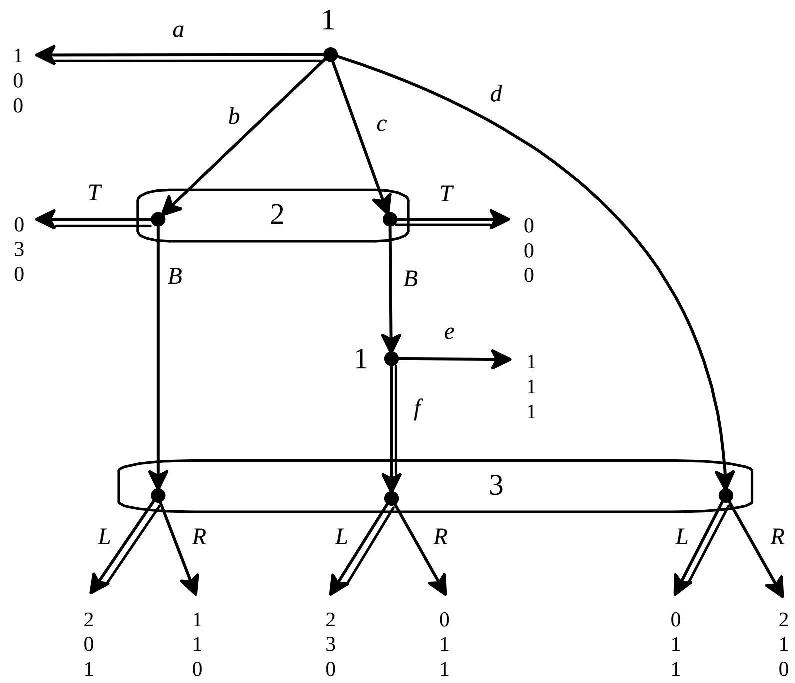


Figure 13.26: The game for Exercise 13.12.

We have to show that the assessment (σ, μ) with

$$\sigma = (a, T, f, L) \qquad \mu = \left(\begin{array}{cc|ccc} b & c & bB & cBf & d \\ \frac{7}{10} & \frac{3}{10} & \frac{7}{18} & \frac{3}{18} & \frac{8}{18} \end{array} \right)$$

is a sequential equilibrium. By Theorem 13.5.1 it is sufficient to show that (σ, μ) is sequentially rational, is rationalized by a choice measurable plausibility order and is uniformly Bayesian relative to it.

First we verify sequential rationality.

- For Player 1 at the root the possible payoffs are: 1 with a, 0 with b, 0 with c (more precisely, with either ce or cf) and 0 with d; thus a is sequentially rational.
- For Player 1 at history cB the possible payoffs are: 1 with e and 2 with f; thus f is sequentially rational.
- For Player 2 the possible payoffs are:
  $\frac{7}{10}(3) + \frac{3}{10}(0) = \frac{21}{10}$ with T
  and $\frac{7}{10}(0) + \frac{3}{10}(3) = \frac{9}{10}$ with B;
  thus T is sequentially rational.
- For Player 3 the possible payoffs are:
  $\frac{7}{18}(1) + \frac{3}{18}(0) + \frac{8}{18}(1) = \frac{15}{18}$ with L
  and $\frac{7}{18}(0) + \frac{3}{18}(1) + \frac{8}{18}(0) = \frac{3}{18}$ with R;
  thus L is sequentially rational.

Next we show that (σ,μ) is rationalized by a choice-measurable plausibility order. It is straightforward to verify that (σ,μ) is rationalized by the plausibility order shown below together with a choice-measurable integer-valued representation F:

$$\left(\begin{array}{ll|c}
 & \precsim : & F: \\
\hline
\text{most plausible} & \emptyset, a & 0 \\
 & b, c, bT, cT & 1 \\
 & d, bB, cB, cBf, dL, bBL, cBfL & 2 \\
\text{least plausible} & dR, bBR, cBfR, cBe & 3
\end{array}\right)$$

To see that (σ,μ) is uniformly Bayesian relative to this plausibility order, let E_1, E_2 and E_3 be the top three equivalence classes of the order and consider the following probability distributions, which satisfy the properties of Definition 13.2.1 (so that (σ,μ) is Bayesian relative to this plausibility order):

$$\nu_{E_1}(\emptyset) = 1, \quad \nu_{E_2} = \begin{pmatrix} b & c \\ \frac{7}{10} & \frac{3}{10} \end{pmatrix} \quad \text{and} \quad \nu_{E_3} = \begin{pmatrix} bB & cB & cBf & d \\ \frac{7}{21} & \frac{3}{21} & \frac{3}{21} & \frac{8}{21} \end{pmatrix}.$$

A full support common prior that satisfies the properties of Definition 13.4.1 is the following:

$$\nu = \begin{pmatrix} \emptyset & b & bB & c & cB & cBf & d \\ \frac{9}{40} & \frac{7}{40} & \frac{7}{40} & \frac{3}{40} & \frac{3}{40} & \frac{3}{40} & \frac{8}{40} \end{pmatrix}.$$

Solutions to Exercise 13.13 The extensive form under consideration is shown in Figure 13.27.

Let $\sigma = (M, \ell, a, c, e)$, $\quad \mu = \left(\begin{array}{ccc|cc|cc|cc} L & M & R & L\ell & Mr & Lm & Rr & Mm & R\ell \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right).$

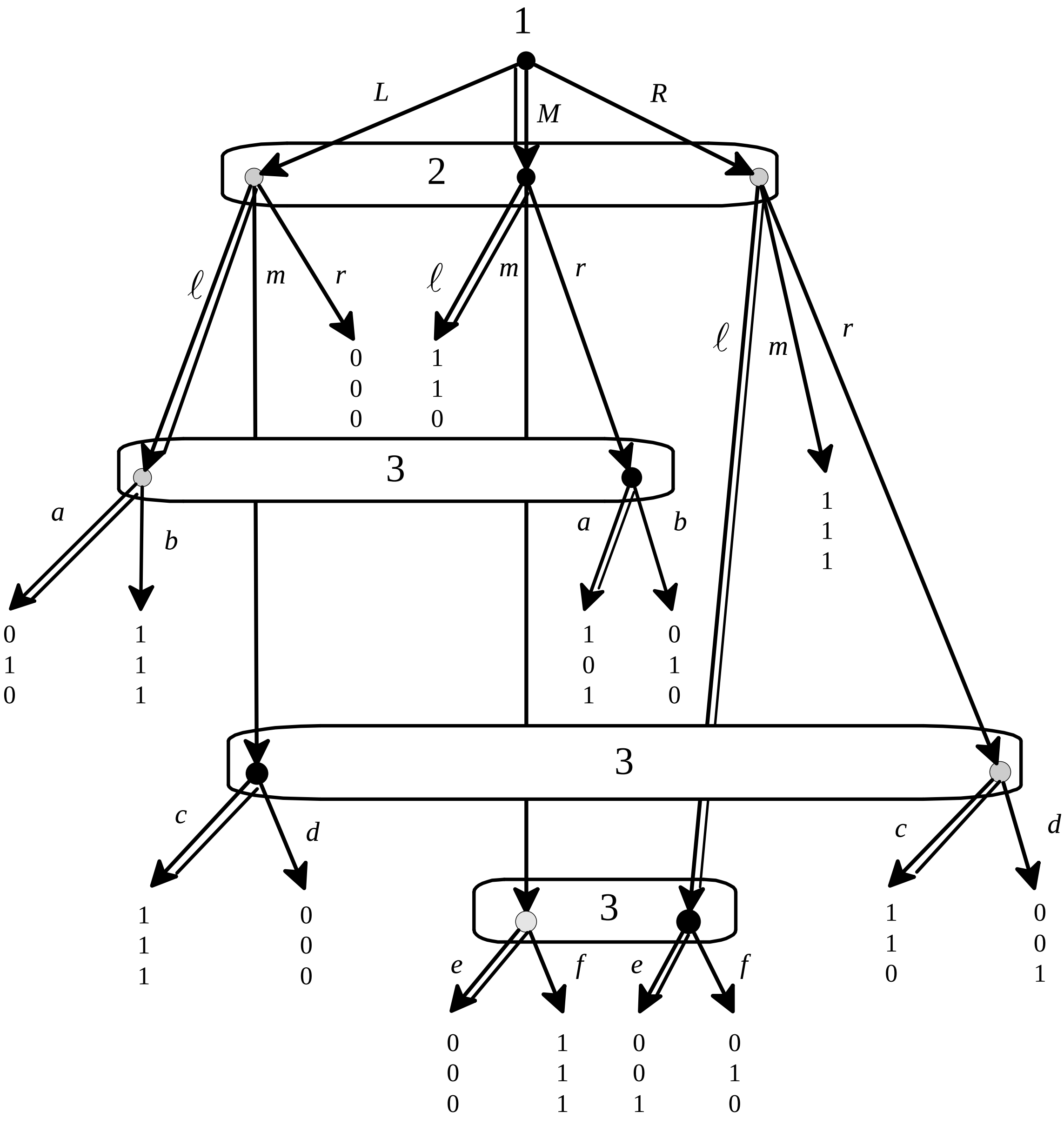


Figure 13.27: The extensive form for Exercise 13.13.

(a) We have to show that (σ, μ) is a strongly independent perfect Bayesian equilibrium. Sequential rationality is straightforward to verify. It is also straightforward to verify that the following plausibility order rationalizes (σ, μ):

$$\left(\begin{array}{ll} \text{most plausible} & \emptyset, M, M\ell \\ & R, R\ell, R\ell e \\ & Mm, Mme \\ & Mr, Mra \\ & L, L\ell, L\ell a \\ & Rm \\ & Lm, Lmc \\ & Rr, Rrc \\ & Lr \\ & R\ell f \\ & Mmf \\ & Lmd \\ & Rrd \\ & Mrb \\ \text{least plausible} & L\ell b \end{array}\right)$$

Let us check that the above plausibility order satisfies properties (IND_1) and (IND_2). For (IND_1) first note that there are no two decision histories h and h' that belong to the same information set and are such that $h \sim h'$; thus we only need to check that if h and h' belong to the same information set and $h \succ h'$ then $ha \succ h'a$ for every $a \in A(h)$. This is indeed true:

1. $M \succ R$ and $M\ell \succ R\ell$, $Mm \succ Rm$ and $Mr \succ Rr$,
2. $M \succ L$ and $M\ell \succ L\ell$, $Mm \succ Lm$ and $Mr \succ Lr$,
3. $R \succ L$ and $R\ell \succ L\ell$, $Rm \succ Lm$ and $Rr \succ Lr$,
4. $Mr \succ L\ell$ and $Mra \succ L\ell a$ and $Mrb \succ L\ell b$,
5. $Lm \succ Rr$ and $Lmc \succ Rrc$ and $Lmd \succ Rrd$,
6. $R\ell \succ Mm$ and $R\ell e \succ Mme$ and $R\ell f \succ Mmf$.

For (IND_2) first note that there is no decision history h that $ha \sim hb$ for $a, b \in A(h)$ with $a \neq b$; thus we only need to show that if $ha \succ hb$ and $h' \in I(h)$ then $h'a \succ h'b$. This is indeed true:

1. $M\ell \succ Mm$ and $L\ell \succ Lm$ and $R\ell \succ Rm$,
2. $Mm \succ Mr$ and $Lm \succ Lr$ and $Rm \succ Rr$,
3. $M\ell \succ Mr$ $L\ell \succ Lr$ and $R\ell \succ Rr$,

and the rest is trivial, since at the other information sets there are only two actions, one of which is plausibility preserving and the other is not.

It remains to show that (σ,μ) is uniformly Bayesian relative to the plausibility order given above. First of all, note that, for every equivalence class E of the order, $E \cap D_\mu^+$ is either empty or a singleton. Thus as a full support common prior one can take, for example, the uniform distribution over the set of decision histories: $\nu(h) = \frac{1}{10}$ for every $h \in \{\emptyset, L, M, R, L\ell, Mr, Lm, Rr, Mm, R\ell\}$.

(b) To prove that (σ,μ) is not a sequential equilibrium it would not be sufficient to prove that the plausibility order given above is not choice measurable (although it is indeed true that it is not choice measurable), because in principle there could be another plausibility order which is choice measurable and rationalizes (σ,μ). Thus we need to show that *any* plausibility order that rationalizes (σ,μ) is not choice measurable. Let $\precsim$ be a plausibility order that rationalizes (σ,μ); then it must satisfy the following properties:

- $Lm \succ Rr$ (because they belong to the same information set and $\mu(Lm) > 0$ while $\mu(Rr) = 0$). Thus if F is any integer-valued representation of $\precsim$ it must be that
$$F(Lm) < F(Rr). \tag{13.10}$$
- $Mr \succ L\ell \sim L$ $\big(Mr \succ L\ell$ because Mr and $L\ell$ belong to the same information set and $\mu(Mr) > 0$ while $\mu(L\ell) = 0$; $L\ell \sim L$ because ℓ is a plausibility-preserving action since $\sigma(\ell) > 0\big)$). Thus if F is any integer-valued representation of $\precsim$ it must be that
$$F(Mr) < F(L). \tag{13.11}$$
- $R \sim R\ell \succ Mm$ $\big(R \sim R\ell$ because ℓ is a plausibility-preserving action; $R\ell \succ Mm$ because $R\ell$ and Mm belong to the same information set and $\mu(R\ell) > 0$ while $\mu(Mm) = 0\big)$. Thus if F is any integer-valued representation of $\precsim$ it must be that
$$F(R) < F(Mm). \tag{13.12}$$

Now suppose that $\precsim$ is choice measurable and let F be an integer-valued representation of it that satisfies choice measurability. From (13.10) and (13.11) we get that

$$F(Lm) - F(L) < F(Rr) - F(Mr) \tag{13.13}$$

and by choice measurability

$$F(Rr) - F(Mr) = F(R) - F(M). \tag{13.14}$$

If follows from (13.13) and (13.14) that

$$F(Lm) - F(L) < F(R) - F(M). \tag{13.15}$$

Subtracting $F(M)$ from both sides of (13.12) we obtain

$$F(R) - F(M) < F(Mm) - F(M). \tag{13.16}$$

It follows from (13.15) and (13.16) that $F(Lm) - F(L) < F(Mm) - F(M)$, which can be written as $F(M) - F(L) < F(Mm) - F(Lm)$, yielding a contradiction, because choice measurability requires that $F(M) - F(L) = F(Mm) - F(Lm)$. □