Game Theory through EXAMPLES

ERICH PRISNER

MAA

CLASSROOM RESOURCE MATERIALS

Game Theory
Through Examples

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Game Theory Through Examples

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Preface

Welcome to game theory, the mathematical theory of how to analyze games and how to play them optimally. Although "game" usually implies fun and leisure, game theory is a serious branch of mathematics. Games like blackjack, poker, and chess are obvious examples, but there are many other situations that can be formulated as games. Whenever rational people must make decisions within a framework of strict and known rules, and where each player gets a payoff based on the decisions of all the players, we have a game. Examples include auctions, negotiations between countries, and military tactics. The theory was initiated by mathematicians in the first half of the last century, but since then much research in game theory has been done outside of mathematics.

This book gives an introduction to game theory, on an elementary level. It does not cover all areas of a field that has burgeoned in the last sixty years. It tries to explain the basics thoroughly, so that it is understandable for undergraduates even outside of mathematics, I hope in an entertaining way. The book differs from other texts because it emphasizes examples. The theory is explained in nine theory chapters, and how it is applied is illustrated in twenty-four example chapters, where examples are analyzed in detail. Each example chapter uses tools developed in theory chapters.

Audience

The text can be used for different purposes and audiences. It can be used by students of economics, political science, or computer science. Undergraduate mathematics students should enjoy the book as well and profit from its approach. The book may also be used as a secondary text, or for independent study, as its many concrete examples complement expositions of the theory.

I think there is another audience that could profit from the book, the one I had in mind when I wrote the book, which is undergraduates who take a mathematics class to fulfill general education or quantitative reasoning requirements. The text began as an online text for a first year seminar at Franklin College in 2007, when I couldn't find an appropriate text. Since then, I have used versions of the text in our first year seminar and in a liberal arts course I have called Introduction to Game Theory. Game theory and my approach to it are well suited for these courses because:

- The underlying mathematics is basic: for example, finding minima and maxima, computing weighted averages and working systematically through trees or digraphs.
- The mathematical topics that are covered—probability, trees, digraphs, matrices, and algorithms—are among the most useful tools used in applications.
- Game theory is applied mathematics. Some would claim it is not even mathematics, but part of economics. The "why are we doing this?" question has an an obvious answer. Students accept the fact that mathematical tools have to be developed for studying and analyzing games.
- Game theory gives opportunities for students to do projects.
- Game theory allows a playful approach.

Features of the book

There are thirty-eight chapters of three types:

- Nine chapters present the basic theory. They are listed as theory chapters in the table of contents. In most courses all these chapters would be covered. There is an additional chapter explaining how to use the Excel files, which is also required reading.
- Five chapters describe notable landmarks in the history and development of game theory: the 1654 letters between Pascal and Fermat, which mark the beginning of probability theory; Princeton University before 1950, where game theory started as an academic discipline; RAND Corporation in the 1950s and the early optimism there about what game theory could achieve; casino games; and the Nobel prizes awarded to game theorists. The chapters provide background about some of the important persons in the development of game theory, and discuss game theory's role in society. The chapters might be for reading at home for students. They could also be vehicles for class discussions.
- The real core of the manuscript are the twenty-four chapters that treat concrete examples. They distinguish the book from others. The chapters use the theory developed in the theory chapters, and they usually build on at most one other example chapter, so an instructor can select those that seem most appropriate. In my classes, I usually discuss about eight examples. Some of the example chapters provide glimpses into areas that are not covered in the theory chapters. Examples of this are trees in Chapter 4, voting power indices in Chapter 7, complexity and binomial coefficients in Chapter 9, statistics, mechanism design, and incomplete information in Chapter 15, incomplete versus imperfect information in Chapter 23, work with parameters and algebra in Chapter 31, and cooperative games in Chapter 35.

There are two other main features of the book: the Excel spreadsheets and Javascript applets.

- The Javascript applets are small games, in which students can try out many of the games discussed and analyzed in the examples. Students can play against the computer or against students. Before reading the analysis students should have played the games and developed some ideas on how to play the game. The Javascripts are also used to confirm theoretical results.
- The Excel spreadsheets either are generally usable for any simultaneous 2-player game or for every 2-player game in normal form, or they are designed for specific games. They give the students tools to check what is described in the text without having to do many routine or tedious calculations, and also tools to apply "what if" analysis to games. Students will see, sometimes in unexpected ways, how changing parameters may change strategies, outcomes, and payoffs.

In my opinion Excel is the best tool for the game theory in this book, better than Mathematica or Maple or MathCad. Since all steps are elementary (no functions are used except, "Min", "Max", "Sum", and "If""), everything could also be done by the student on a sheet of paper, at least in principle. Every student should learn some Excel anyway, another feature which makes a course based on this book very well-suited for general education core requirements.

For the Instructor

The book's presentation is informal. It emphasizes not formulas but understanding, interpretation, and applicability. In my opinion, formalism and proof are perfect tools and unavoidable for professional mathematicians but are of limited value for students, even to some extent for undergraduate students of mathematics. Some proofs of theorems are given, for others informal reasonings, but often students have to rely on evidence from the examples and simulations given. I try to avoid using complicated formulas. When I present a formula, I try to explain it in words. It helps students to concentrate on concrete examples with concrete numbers. When the book introduces parameters, the analysis carries them forward as a first step towards abstraction.

Although the book is elementary and avoids formalism, it may not be easy reading. Some books consider only 2 by 2 bimatrix games or simple real-life situations, but the examples in this book are sometimes complex. In my opinion, the power of mathematics cannot be appreciated by looking only at small examples. In them little mathematics is needed and common sense shows the best strategy. The students who use this book will use tables of numbers with ten to twenty rows and columns. Although this requires more effort than smaller examples, the Excel sheets will help with the calculations, and in the end the results should justify the extra effort. Students will see that mathematical theory sometimes produces unexpected results.

Though the examples are sometimes complex, they are still far removed from most real games or from serious modeling. I try to make this clear throughout the book. That does not mean that no lessons can be drawn from them!

It is common knowledge that mathematics can be learned only by doing. The book contains sixty or so projects that ask for the solution of a game and the presentation of it. Experimenting and going beyond what is described in the text is in my opinion crucial for mastering the material. Some projects are difficult and some are open-ended. I tell my students to do as much as they can, and to do that well!

Figure 1 shows how the chapters are related. The eight theory chapters are squares, the Excel is a diamond, the five history chapters are stars, and the example chapters are circles. The backbone of every course would be the theory chapters and the Excel chapter, chapters 1, 2, 6, 8, 12, 16, 22, 24, 27, and 32. They should in my opinion be covered in every course on game theory. Some of the history chapters, chapters 13, 18, 21, 28, and 39, could be assigned as reading and material for class discussion.

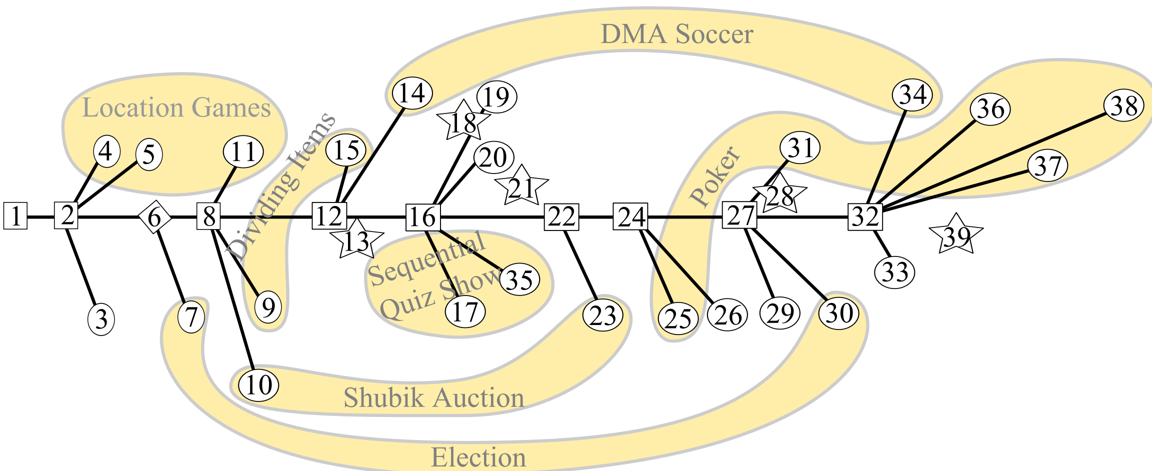

Figure 1. Structure of the chapters

Each example chapter is attached by a line to the earliest theory chapter after which it can be covered. Some of the example chapters form pairs, like Election and Election II, which are enclosed by dashed lines. I would not recommend covering a part II chapter without having covered part I. There are two larger groups, the location games of chapters 4, 5, and 11, and the poker chapters 25, 31, 36, 37, 38. In these cases I would recommend at least one of chapters 4 or 5 before chapter 11, and covering chapter 25 before covering any of chapters 31, 36, 37, or 38.

As more tools become available, the example chapters become more complex and difficult, so later chapters are generally more difficult than early ones. In Figure 1 I indicate the difficulty of the example chapters by showing them above or below the theory backbone. The example chapters below the backbone I consider to be easier, those above more difficult. But sometimes it is possible to skip the difficult parts. For example, chapters 4, 5, and 11 all contain proofs that relate games to the structure of graphs, and this may be of interest to mathematics or computer science majors. But the games provide all readers with instructive examples of simultaneous and sequential games.

When I teach my course, I usually cover at least one example chapter after each theory chapter. For a general education mathematics course an appropriate set may be chapters 3, 4 (just Section 4.1), 7, 10, 17, 23, 26, 30, and 35.

Though the poker chapters 31, 36, and 37, are complex, I usually cover one of them in my course, since I end the semester with a robot poker tournament similar to the one described in chapter 38. The tournament gets the student's attention and is usually a lot of fun! Instructors can modify the applets to run their own tournaments and even vary the rules of the game.

Acknowledgements

Early versions of this book were used as an online text for seven classes at Franklin College (now Franklin University) between 2007 and 2013. I want to thank all my students for fighting their way through these often incomplete and imperfect pages, and for the valuable feedback they gave me. I want to thank in particular Samy Al-Ansari, Nurbek Almakuchukov, Meghan Canale, Eric Gors, Ruth Kappes, Matej Kristic, Grigore Tocitu, Karam Zaidkilani, Conner Crawford, Chikara Davis, James Jasper, Artyom Machekin, Nicolaus Owens, Megan Austin, Dylan Bontempo, Maija Butler, Andriana Friel, Rene Musech, Sannan Panaruna, Aaron Brinckerhoff, Caitlin Curtis, Mihael Djambazov, Mariah Grubb, Sophie Winckel, Nathan Ayers, Jennalee Bauman, Alex Bromwell, Stephanie Erichsen, Sachint Goel, Segev Goldberg, Andrea Hak, Michael Radomile, Cooper Stoulil, Jennifer Byram, Arnulfo Hermes, Connor McCormack, Benjamin Smick, Paige Stumbough, Emil Akhmetov, (Kevin) Yikfai Chan, Todd Gilbert, Ishan Hill, Anna Hixson, Adrian Lewis, Ariane Mottale, Lily Rybarczyk, Adam Taylor, and Senya Webster.

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CHAPTER 1

Theory 1: Introduction

1.1 What's a Game?

Every child understands what games are. When someone overreacts, we sometimes say "it's just a game." Games are often not serious. *Mathematical* games, which are the subject of this book, are different. It was the purpose of game theory from its beginnings in 1928 to be applied to serious situations in economics, politics, business, and other areas. Even war can be analyzed by mathematical game theory. Let us describe the ingredients of a mathematical game.

Rules Mathematical games have strict rules. They specify what is allowed and what isn't. Though many real-world games allow for discovering new moves or ways to act, games that can be analyzed mathematically have a rigid set of possible moves, usually all known in advance.

Outcomes and payoffs Children (and grown-ups too) play games for hours for fun. Mathematical games may have many possible **outcomes**, each producing **payoffs** for the players. The payoffs may be monetary, or they may express satisfaction. You want to win the game.

Uncertainty of the Outcome A mathematical game is "thrilling" in that its outcome cannot be predicted in advance. Since its rules are fixed, this implies that a game must either contain some random elements or have more than one player.

Decision making A game with no decisions might be boring, at least for the mind. Running a 100 meter race does not require mathematical skills, only fast legs. However, most sport games also involve decisions, and can therefore at least partly be analyzed by game theory.

No cheating In real-life games cheating is possible. Cheating means not playing by the rules. If, when your chess opponent is distracted, you take your queen and put it on a better square, you are cheating, as in poker, when you exchange an 8 in your hand with an ace in your sleeve. Game theory doesn't even acknowledge the existence of cheating. We will learn how to win without cheating.

1.2 Game, Play, Move: Some Definitions

The complete set of rules describes a **game**. A **play** is an instance of the game. In certain situations, called **positions**, a player has do make a decision, called a **move** or an **action**. This is not the same as strategy. A **strategy** is a plan that tells the player what move to choose in every possible position.

Rational behavior is usually assumed for all players. That is, players have preferences, beliefs about the world (including the other players), and try to optimize their individual payoffs. Moreover, players are aware that other players are trying to optimize their payoffs.

1.3 Classification of Games

Games can be categorized according to several criteria:

- How many players are there in the game? Usually there should be more than one player. However, you can play roulette alone—the casino doesn't count as player since it doesn't make any decisions. It collects or gives out money. Most books on game theory do not treat one-player games, but I will allow them provided they contain elements of randomness.
- Is play simultaneous or sequential? In a **simultaneous game**, each player has only one move, and all moves are made simultaneously. In a **sequential game**, no two players move at the same time, and players may have to move several times. There are games that are neither simultaneous nor sequential.
- Does the game have random moves? Games may contain random events that influence its outcome. They are called **random moves**.
- Do players have perfect information? A sequential game has **perfect information** if every player, when about to move, knows all previous moves.
- Do players have **complete information**? This means that all players know the structure of the game—the order in which the players move, all possible moves in each position, and the payoffs for all outcomes. Real-world games usually do not have complete information. In our games we assume complete information in most cases, since games of incomplete information are more difficult to analyze.
- Is the game zero-sum? **Zero-sum games** have the property that the sum of the payoffs to the players equals zero. A player can have a positive payoff only if another has a negative payoff. Poker and chess are examples of zero-sum games. Real-world games are rarely zero-sum.
- Is communication permitted? Sometimes communication between the players is allowed before the game starts and between the moves and sometimes it is not.
- Is the game cooperative or non-cooperative? Even if players negotiate, the question is whether the results of the negotiations can be enforced. If not, a player can always move differently from what was promised in the negotiation. Then the communication is called "cheap talk". A **cooperative game** is one where the results of the negotiations can be put into a contract and be enforced. There must also be a way of distributing the payoff among the members of the coalition. I treat cooperative games in Chapter 35.

Student Activity Play ten rounds in the applets for each one of the games `LisaGame`, `QuatroUno`, and `Auction`. In the first two you can play against a (well-playing) computer, but in the third the computer serves only as auctioneer and you need to find another human player. Categorize each game as simultaneous or sequential or neither, and determine whether randomness is involved, whether the game has perfect information, and whether it is zero-sum. Determine the number of players in each game, and justify your conclusions.

Modeling Note Analyzing games like parlor games or casino games may seem to be enough motivation to develop a theory of games. However, game theory has higher aims. It provides tools that can be applied in many situations where two or more persons make decisions influencing each other.

A **model** is an abstract, often mathematical, version of reality. In this book a model is a game, which is supposed to yield some insight into a real-world situation. It is important not to confuse the model with reality—in reality there are almost never totally strict rules and players almost always have more options than they think they have, more than what the model allows.

In this book we also will try to model some real-world situations as games, but the approach taken is cautious. Whenever we try to model a real-life situation, we will discuss in detail the assumptions of the model and whether the conclusions from the model are relevant. Whether game theory can be useful in real life is something for each reader to decide.

Exercises

1. In English auction, an item is auctioned. People increase bids in increments of \$10, and the player giving the highest bid gets the item for that amount of money. Give reasons why the auctioneer would be considered a player of the game, or reasons why he or she would not. Does the game contain random moves? Is it zero-sum? Would a real-world art auction have complete information?
2. In roulette, would the croupier be considered to be a player? Does the game contain random moves? Is it zero-sum? Can players increase their chances of winning if they form a coalition and discuss how to play before each round?
3. In the well-known game rock, scissors, paper game, how many players are there? Is it simultaneous, or sequential, or neither, and, if it is sequential, does it have perfect information?
4. For poker, discuss number of players, whether it is sequential or simultaneous, or neither, and if it is sequential, whether it has perfect information. Discuss whether there are random moves. Is communication allowed in poker?
5. For blackjack discuss its number of players; whether it is sequential or simultaneous, or neither; and if it is sequential, whether it has perfect information. Discuss whether there are random moves. Is communication allowed in blackjack?
6. It's late afternoon and you are in a train traveling along a coastline. From time to time the train stops in villages, some of them nice, some of them ugly, and you can evaluate the niceness of the village immediately. The benefit of an evening and night spent at that village depends only on its niceness. You want to get off at the nicest village. Unfortunately you don't know how many villages are still to come, and you know nothing about how villages in this country normally look. Worse, you are not able to ask anybody, since you don't speak the language of the country. You also know that some (unknown) time in the evening the train will reach its destination where you will have to stay whether it is nice or not. Explain the features of this game, with emphasis on the informational issues. How would you play it? Give a reason for your strategy. Comment on whether we have complete or incomplete information here, and why.

   (Initially I formulated this example in terms of marriage in a society where divorce is impossible, but I saw that this is a different game. Could you give some arguments why?)
7. In this more realistic version of the game suppose that you know that the train will stop in ten villages before it reaches its destination. How would you play now? Comment on whether we have complete or incomplete information here, and justify your comment.

CHAPTER 2

Theory 2: Simultaneous Games

In his story "Jewish Poker" the writer Ephraim Kishon describes how a man called Ervinke convinces the narrator to play a game called Jewish Poker with him. "You think of a number, I also think of a number", Ervinke explains. "Whoever thinks of a higher number wins. This sounds easy, but it has a hundred pitfalls." Then they play. It takes the narrator some time until he realizes that it is better to let Ervinke tell his number first. [K1961] Obviously this is a game that is not fair unless both players play simultaneously.

In this chapter we will start our journey through game theory by considering games where each player moves only once, and moves are made simultaneously. The games can be described in a table (called the game's **normal form**). Then we discuss approaches that allow the players to decide which move they will choose, culminating with the famous Nash equilibrium.

2.1 Normal Form—Bimatrix Description

Imagine you want to describe a simultaneous game. We know that each player has only one move, and that all moves are made simultaneously. What else do we need to say? First, we must stipulate the number of players in the game. Second, we must list for each player all possible moves. Different players may have different roles and may have different options for moves. We assume that each player has only finitely many options. Players simultaneously make their moves, determine the **outcome** of the game, and receive their payoffs. We need to describe the payoffs for each outcome.

How many outcomes are possible? Each combination of moves of the players generates a different outcome. If there are n players, and player 1 has k_1 possible moves, player 2 has k_2 possible moves, and so on, then there are $k_1 \times k_2 \times \cdots \times k_n$ possible outcomes. For each, n numbers would describe the payoffs for player 1, player 2, and so on.

In games where each player has infinitely many options, we may use methods of calculus for functions with two variables, but such games are not discussed in this book. We describe simultaneous games with randomness in Chapter 12.

2.1.1 Two Players

Here is an example of a simultaneous 2-player game:

Example 1 **ADVERTISING:** Two companies share a market, in which they currently make \$5,000,000 each. Both need to determine whether they should advertise. For each company advertising costs \$2,000,000 and captures \$3,000,000 from the competitor provided the competitor doesn't advertise. What should the companies do?

Let's call the two companies A and B. If both don't advertise, they get $5,000,000 each. If both advertise, both lower their gain to $3,000,000. If A advertises, but B doesn't, A gets $6,000,000 and B only $2,000,000, and conversely if B advertises and A doesn't. The payoff pattern is shown in the following table. The numbers are in millions of dollars. The rows correspond to the options of player A, and the columns correspond to the options of player B. The entries are payoff for A and payoff for B provided the corresponding options are chosen, separated by a comma.

	B advertises	B doesn't advertise
A advertises	3, 3	6, 2
A doesn't advertise	2, 6	5, 5

Whenever we have two players, we often name them Ann and Beth. Assume Ann has k_1 options and Beth has k_2 options. We want to display the different payoffs for Ann and Beth, depending on the different choices they have. Each of the $k_1 \cdot k_2$ outcomes has payoffs for Ann and Beth attached. Usually this is visualized in a table, the **normal form** with n rows, corresponding to Ann's options, and m columns, corresponding to Beth's options. Such a table is called a $n \times m$ bimatrix. The entries in the cells are payoffs for Ann and Beth, separated by a comma.

2.1.2 Two Players, Zero-sum

A game is called **zero-sum** if the sum of payoffs equals zero for any outcome. That means that the winnings of the winning players are paid by the losses of the losing players.

For zero-sum two-player games, the bimatrix representation of the game can be simplified: the payoff of the second player doesn't have to be displayed, since it is the negative of the payoff of the first player.

Example 2 Assume we are playing ROCK-SCISSORS-PAPER for one dollar. Then the payoff matrix is

	Rock	Scissors	Paper
Rock	0	1	−1
Scissors	−1	0	1
Paper	1	−1	0

The first cell says "0", which stands for "0, 0" a payoff of 0 for both players. The second cell entry of "1" should be read as "$1, -1$", a payoff of 1 for Ann which has to be paid by Beth, therefore a payoff of -1 for Beth.

2.1.3 Three or More Players

If we have more than two players, we need another systematic way to generate the needed $k_1 \cdot k_2 \cdots k_n$ cells corresponding to the different outcomes, into which we write the n payoffs for the n players. Here is an example:

Example 3 **LEGISLATORS' VOTE:** Three legislators vote whether they allow themselves a raise in salary of $2000 per year. Since voters are observing the vote, there is some loss of face for a legislator to vote for a raise. Let's assume that the legislators estimate that loss of face is worth $1000 per year. What happens if all three vote at the same time? (This game is a variant of the game described in [K2007]).

This is a simultaneous three-player game. It is best visualized with two matrices. Player A chooses the matrix, B chooses the row, and C chooses the column. The payoffs (in thousands of dollars) are

A votes for a raise

	C votes for raise	C votes against it
B votes for raise	1, 1, 1	1, 1, 2
B votes against	1, 2, 1	-1, 0, 0

A votes against a raise

	C votes for raise	C votes against it
B votes for raise	2, 1, 1	0, -1, 0
B votes against	0, 0, -1	0, 0, 0

2.1.4 Symmetric Games

All our examples so far are **symmetric**: All players have the same options, and if the two players interchange their moves, the payoffs are also interchanged. More formally, for a 2-player game, let m_1, m_2 be moves and let $a(m_1, m_2)$ and $b(m_1, m_2)$ be Ann's and Beth's payoffs if Ann plays m_1 and Beth plays m_2. Then $a(m_1, m_2) = b(m_2, m_1)$ and $b(m_1, m_2) = a(m_2, m_1)$ for symmetric games. That means that the entries in the js row and the is column is obtained from the entries in the is row and js column by interchanging the payoffs. For symmetric 3-player games, $a(m_1, m_2, m_3) = b(m_2, m_1, m_3) = b(m_3, m_1, m_2) = c(m_2, m_3, m_1) = c(m_3, m_2, m_1)$, and so on. Symmetric games are fair by design, giving the same chances to every player.

2.2 Which Option to Choose

It is useful to describe simultaneous games by a bimatrix, but what players want is advice on how to play. Game theory should (and will in some cases) provide players with a mechanism to find which move is best.

The mechanisms would refer to the bimatrix only, the solution would be the same no matter whether we face a casino game or a war, provided the corresponding matrices are the same. The essence of the game lies in the numbers in the bimatrix.

Such mechanisms are the content of this section. We will give three or four of them. Like advice from well-meaning uncles, they all have a good and convincing point, but since they concentrate on different features of the game, they don't always lead to the same conclusion. We will discuss them first separately before investigating the relations between them.

2.2.1 Maximin Move and Security Level

Some people always expect the worst. No matter what she plays, a player (let's call her Ann) may assume that the other players will always respond with moves that minimize Ann's payoff. This may be justified in a two-player zero-sum game if Ann is so predictable that the other player always anticipate her move. In other cases the belief borders on paranoia, since the other players will not be interested in harming Ann but instead want to maximize their payoffs. Still, pessimistic Ann will evaluate her strategies in light of the worst expected case. She would concentrate, for any of her options, on her smallest possible payoff. If she believes that this is what she would get, then Ann would choose the option with highest value. This value is called the **maximin value** or **security level**. The option Ann will play is called a **maximin move (strategy)**, since it maximizes the minimum possible payoff. Playing the maximin move, the player can guarantee a payoff of at least the maximin value, no matter how the others are playing. To choose the maximin move, the player doesn't have to know the payoffs of the other players.

In the ADVERTISING example, company A may fear that company B will advertise too if A advertises, yielding a payoff of 3 for A. If company A does not advertise, the worst that could happen would be company

B advertising with payoff of 2 for A. Therefore company A would advertise to maximize the worst possible payoff.

In the LEGISLATORS' VOTE example, the worst that could happen if A votes for a raise is that both others vote against, leaving A with a payoff of -1000. If A votes against a raise, in the worst case (actually in three of four cases) A gets a payoff of 0, which is more than in the other case. Therefore A would vote against a raise if using the maximin principle.

In a two-player game the first player, Ann, would look at the rows of the bimatrix and in each row highlight the cell with her lowest payoff. Then she would select the row with the highest number highlighted. In the same way, the second player, Beth, when playing the maximin strategy would mark in each column the cell with lowest payoff for Beth, and then select the column with the highest number marked.

How do we treat ties, if two or more rows have the same minimum payoff for Ann? Ann could choose one of these moves, or alternate randomly between such moves. The latter leads to mixed strategies that are covered in Chapter 27.

2.2.2 Dominated Moves

A move, $M1$, for Ann **strictly dominates** another $M2$, if $M1$ always results in a higher payoff for Ann than $M2$. A rational player would never play a move that is strictly dominated by another one. Domination doesn't tell what to play but rather what not to play. In the rare case where one of Ann's moves strictly dominates all her other moves, this would turn into positive advice to play the move dominating all other moves.

In the ADVERTISING example "advertising" strictly dominates "not advertising" for both companies. Therefore both companies will advertise, when applying this mechanism.

It is no coincidence that the advice given by the maximin mechanism and the advice given by the rule not to play strictly dominated moves are the same for this example. Actually a player's maximin move is never strictly dominated by any of her other moves.

Advice for players could go further than to disregard strictly dominated moves. In particular, if player Ann believes that other players would also obey this rule, then we may disregard all strictly dominated moves in the game, not only for Ann but for all other players. However, this assumption about the other players' behavior is not automatic. It assumes that all players are rational and clever or experienced enough. Under the assumption that all players accept this belief in the rationality and sophistication of all players, we know that all players reduce the game by eliminating all strictly dominated moves. Then, in the reduced game, strict domination may occur where it had not before, and the same round of eliminations could be done to reduce the game further. The process of repeatedly reducing the game, as well as its result, a game that cannot be reduced any further since there are no strictly dominated moves, is denoted by **IESD—iterated elimination of strictly dominated moves**. Except in cases where the IESD result is a game with just one option for Ann, IESD is a method of excluding moves rather than telling what move to choose.

Here is a successful application of the IESD procedure:

Example 4 TWO BARS: Each one of two bars charges its own price for a beer, either \$2, \$4, or \$5. The cost of obtaining and serving the beer can be neglected. It is expected that 6000 beers per month are drunk in a bar by tourists, who choose one of the two bars randomly, and 4000 beers per month are drunk by natives who go to the bar with the lowest price, and split evenly in case both bars offer the same price. What prices would the bars select? [S2009]

The game, as all games considered so far, is symmetric. Let me illustrate in one instance how to compute the payoffs. If bar A charges \$2 and bar B charges \$4, then all natives will choose bar A.

Therefore bar A will serve 4000 beers to the natives, and 3000 beers to tourists, serving 7000 beers in total, making $7000 \cdot 2 = 14000$ dollars. Bar B will only serve 3000 beers to tourists, making $3000 \cdot 4 = 12000$ dollars.

The payoff matrix, with values in thousands of dollars, is

	2	4	5
2	10, 10	14, 12	14,15
4	12, 14	20, 20	28, 15
5	15, 14	15, 28	25, 25

For each bar, move "4" strictly dominates move "2", therefore we could eliminate both moves "2" to get the reduced game:

	4	5
4	20, 20	28, 15
5	15, 28	25, 25

Now, but not before the elimination, move "4" strictly dominates move "5". Therefore we eliminate these moves for both players as well and arrive at a game with only one option, "4", for each player, and a payoff of \$ 20000 for each. Therefore both players will choose \$4 as the price of the beer.

A weaker condition is weak domination. Ann's move **weakly dominates** another one of her moves if it yields at least the same payoff for Ann in all cases generated by combinations of moves of the other players, and in at least one case an even better payoff. So the weakly dominating move is never worse than the weakly dominated one, and sometimes it is better. The common wisdom is that **iterated elimination of weakly dominated moves, IEWD** is not something that should performed automatically. Weakly dominated moves may still be played, in particular in cases where the weakness of the weakly dominated move appears in a combination with other player's moves that are known not to be played by them. This opinion is also based on different behavior of Nash equilibria (discussed in Section 2.4) under IEWD and IESD.

2.2.3 Best Response

Assume you will play a one-round simultaneous game against your friend tomorrow. Your friend has been thinking about her move, arrives on a decision what move to play, and writes it on a piece of paper so as not to forget it. You get a look at this paper without your friend noticing it. The game thus changes from simultaneous to sequential with perfect information. The move you play under these conditions is called the best response to the move of your friend.

Let us start with two players. Ann's **best response** to Beth's move M is the move that yields the highest payoff for Ann, given Beth's move M. There may be several best responses to a given move. To find Ann's best response to Beth's move M, we don't even have to know Beth's payoffs.

You find the best responses for the first player's (Ann's) moves by looking at the rows of the bimatrix one by one and selecting in each row the cell where the second entry is a maximum. The label of the corresponding column is the best response to the move corresponding to that row. In the same way, to find best responses against the second player's (Beth's) moves we consider the columns and pick in each column the cell with maximum first entry. The label of the corresponding row is the corresponding best response for the move corresponding to that column.

In the ADVERTISING example, the best response to advertising is to advertise, and the best response to not advertising is also to advertise. This holds for both players, since the game is symmetric.

In the TWO BARS example, the best response to a price of "2" is a price of "5", the best response to a price of "4" is a price of "4", and the best response to a price of "5" is a price of "4". The game is symmetric.

Example 5 Let us give an asymmetric example. Assume Ann has four moves, A_1, A_2, A_3, A_4, and Beth has three B_1, B_2, and B_3. The payoff bimatrix of this non zero-sum two-person game is

	B_1	B_2	B_3
A_1	$1, \underline{3}$	$2, 2$	$1, 2$
A_2	$\underline{2}, \underline{3}$	$2, \underline{3}$	$2, 1$
A_3	$1, 1$	$1, \underline{2}$	$\underline{3}, \underline{2}$
A_4	$1, 2$	$\underline{3}, 1$	$2, \underline{3}$

We find Beth's best response to Ann's move A_1 by finding the largest second value (Beth's payoff) in the first row, which is underlined. That implies that Beth's best response to Ann's move A_1 is move B_1. In the same way we underline the highest second values in other rows, and conclude that Beth's best responses to Ann's move A_2 are both moves B_1 and B_2, Beth's best responses to move A_3 are both moves B_2 and B_3, and that Beth's best response to move A_4 is move B_3.

To find Ann's best responses, we underline in each column the highest first (Ann's payoff) entry. Therefore Ann's best response to Beth's move B_1 is A_2, Ann's best response to B_2 is A_4, and Ann's best response to B_3 is A_3.

Best Response for Three Players

Best responses make also sense for games with three or more players. For detecting the best response moves of Beth, we look at the second entries (Beth's payoffs) in each column and mark the highest value. To detect the best response moves for the third player (let's call her Cindy) we look at the third entries of the rows and mark in each row the highest entry. For Ann the method is a little more complicated to explain. Here we look at first entries only, and compare cells having the same position in the different matrices, as "upper left", for instance.

Example 6 Let's find best responses in an example of a simultaneous three-person game where each player has two options, Ann has the moves A_1 and A_2, Beth has B_1 and B_2, and Cindy has C_1 and C_2. Assume the payoffs are

A_1

	C_1	C_2
B_1	$0, \underline{2.1}, 0$	$-1, \underline{1.1}, \underline{0.1}$
B_2	$\underline{1}, 0, -1$	$0, 1, \underline{1.1}$

A_2

	C_1	C_2
B_1	$\underline{0.1}, \underline{1.1}, \underline{1}$	$\underline{1.1}, 0.1, -0.9$
B_2	$-0.9, 1, 0$	$\underline{0.1}, \underline{2}, \underline{0.1}$

Because the highest second entry in the first column is 2.1, it is underlined. The highest second entry in the second column is 1.1, in the third column (first column of the second matrix) 1.1, and in the fourth column 2, so they are underlined. For Cindy's best responses, the highest third entry in the first row of the first matrix is 0.1. The highest third entry in the second row of the first matrix is 1.1. For the second matrix, the highest third entry in the first row is 1, and in the second row 0.1. For Ann, the highest first entry of upper-left cells in the two matrices is 0.1, the highest first entry of upper-right cells is 1.1, and we get 1 respectively 0.1 for the lower-left respectively lower-right cells.

2.2.4 Nash Equilibria

In this section we will identify outcomes—combinations of moves for each player— that are more likely to occur than others. An outcome is called a pure Nash equilibrium provided nobody can gain a higher payoff by deviating from the move, when all other players stick to their choices. A higher payoff for a player may be possible, but only if two or more players change their moves. An outcome, a combination of moves, is a **pure**

Nash equilibrium if each move involved is the best response to the other moves. A cell in the normal form is a pure Nash equilibrium if each entry is marked (underlined in our examples) as being the best response to the other moves. Nash equilibria were introduced by John Nash around 1950.

Nash equilibria are self-enforcing agreements. If some (non-binding) negotiation has taken place before the game is played, each player does best (assuming that the other players stick to the agreement) to play the negotiated move.

In the first half of the book, Nash equilibria will be pure. Chapter 27 will introduce mixed Nash equilibria.

In Example 5, there are two Nash equilibria: (A_2, B_1) and (A_3, B_3). In the symmetric TWO BARS example (4, 4) is the unique pure Nash equilibrium. As another example we consider the famous PRISONER'S DILEMMA.

PRISONER'S DILEMMA Adam and Bob have robbed a bank and been arrested. They are interrogated separately. Adam and Bob have the option to confess (move C) or to remain silent (move S). The police have little evidence, and if both remain silent they will be sentenced to one year on a minor charge. Therefore the police interrogators propose a deal: if one confesses while the other remains silent, the one confessing goes free while the other is sentenced to three years. However, if both talk, both will still be sentenced to two years. If each player's payoff is 3 minus the number of years served in jail, we get the following payoff bimatrix:

	S	C
S	2, 2	0, 3
C	3, 0	1, 1

It seems obvious that both should remain silent, but that's not likely to happen. Each player's move C strictly dominates move S. Furthermore, the best response to move S is C, and the best response to move C is also move C, therefore the pair (C, C)—both confessing forms the unique Nash equilibrium of this game.

The choice C—confessing—with payoffs of only 1 may seem counterintuitive if negotiations can take place in advance, but their terms are non-binding and cannot be enforced. It would be useless to agree on move S in advance, since each of the players would feel a strong urge to deviate (cheat). Only if binding agreements are possible, would both agree on the S-S combination, reaching a higher payoff. Thus PRISONER'S DILEMMA gives a paradoxical result. Players will play moves that result in lower payoffs for both than are possible. This is in part because the rules of the game do not allow binding agreements.

Not every simultaneous game has a (pure) Nash equilibrium. An example is Example 2 ROCK-SCISSORS-PAPER.

Next we consider a game with more than one Nash equilibrium:

Example 7 **BATTLE OF THE SEXES:** A couple, Adam and Beth, decide independently whether to go to a soccer game or to the ballet in the evening. Each person likes to do something together with the other, but the man prefers soccer, and the woman prefers ballet.

To simplify the game, we assume that the total payoff for each player is the sum of the payoffs (in terms of satisfaction) of being at the preferred place, which gives a satisfaction of c satisfaction units, and being together with the partner, giving d satisfaction units. We have two variants, depending on whether c or d is larger, the low or high love variants. The payoff here is satisfaction instead of money. The assumption of the additivity of satisfaction is severe—satisfaction could just as well be multiplicative, or some more complicated function of c and d. It could even be that satisfaction in one area could interfere with satisfaction in the other. The satisfactions may differ for both persons, one appreciating the presence of the other more than the other, or one having a clear preference for soccer or ballet, when the other is indifferent. Examples will be given in the exercises.

As this example was devised before there were cell phones, we assume that no previous communication is possible. Here are the payoff bimatrices for both variants, where Adam chooses the rows and Beth chooses the columns.

High Love version, $c = 1, d = 2$

	soccer	ballet
soccer	3, 2	1, 1
ballet	0, 0	2, 3

Low Love version, $c = 2, d = 1$

	soccer	ballet
soccer	3, 1	2, 2
ballet	0, 0	1, 3

The high love version of BATTLE OF THE SEXES has two Nash equilibria: (soccer, soccer) and (ballet, ballet). For Adam choosing “soccer”, Beth’s best response is “soccer”. For Adam choosing “ballet”, Beth’s best response is “ballet”. Also, Adam’s best response for Beth choosing “soccer” is “soccer”, and his best response for Beth choosing “ballet” is “ballet”. The low love version has one Nash equilibrium, namely (soccer, ballet): both players go where they want to go anyway.

Modeling Note We made a simple assumption in the example, namely that the total payoff for a player is the sum of the utilities of certain ingredients. In many situations we will use this approach, since it is simple and is the way money is added. However, there are situations where additivity is not appropriate. One asset may establish worth only when combined with another asset, as a left shoe and the corresponding right shoe, or money and free time available. In many situations each has real value only in combination with the other.

Games with more than one pure Nash equilibrium are sometimes called “coordination games”, since if pregame negotiations are allowed, the players have to agree on one of them. The high love version of BATTLE OF THE SEXES is an example. In this case, the obvious question is: which Nash equilibrium is the best? One idea is to concentrate on Pareto-optimal Nash equilibria only. A Nash equilibrium is **Pareto-dominated** by another Nash equilibrium if every player’s payoff in the first one is smaller or the same as in the second one. Nobody would object to move to the second Nash equilibrium. A Nash equilibrium is **Pareto-optimal** if it is not Pareto-dominated by any other Nash equilibrium, except maybe by some having exactly the same payoffs. In the BATTLE OF THE SEXES example, both Nash equilibria are Pareto-optimal.

For games with more than two players, we use the marking (underlining) procedure as described in the section on best responses. Then the cells with all entries underlined are the pure Nash equilibria.

In Example 6, a 3-player game, we have two pure Nash equilibria—the cells where all entries are underlined, where each move is the best response to the pair of moves of the other two players. These are the triples (A_2, B_1, C_1) and (A_2, B_2, C_2). So player A will probably choose A_2.

In our other example of a 3-player game, LEGISLATORS' VOTE, let us underline the best responses:

	A votes for a raise		A votes against a raise	
	C votes for raise	C votes against it	C votes for raise	C votes against it
B votes for raise	1, 1, 1	$\underline{1}, \underline{1}, \underline{2}$	$\underline{2}, \underline{1}, \underline{1}$	$0, -1, 0$
B votes against	$\underline{1}, \underline{2}, \underline{1}$	$-1, 0, 0$	$0, 0, -1$	$\underline{0}, \underline{0}, \underline{0}$

Here we have four pure Nash equilibria: the three outcomes where two legislators vote for a raise and one votes against, and the one where all three vote against. The fourth equilibrium is Pareto-dominated by the other three, so it is not Pareto-optimal and is therefore less important than the other three.

The next example, a 5-player game, illustrates how you can determine whether an outcome is a Nash equilibrium when you don't have a bimatrix representation:

Example 8 **5 KNIGHTS:** Five knights, A, B, C, D, E, are electing their leader. Each one has a list of preferences. Examples of preferences, given from highest to lowest, are

A: A, D, E, C, B

B: B, C, E, A, D

C: C, E, D, B, A

D: D, B, C, E, A

E: E, C, B, A, D.

They elect in rounds. In each round, each knight submits one name. A knight is elected if he gets more votes than all the others. So even two votes may suffice if no other knight gets two votes. If no one is elected, we proceed to the next round. There are two versions:

Early Case If the knight's first choice is elected, this is a payoff of 2 for that knight. If his second choice is elected, his payoff is 1. If nobody is elected and we proceed to the next round, the payoff is 0. If his third, fourth, or fifth choice is elected, his payoff is -1, -2, or -3.

Exhausted Case The knight's first, second, and third choice gives payoffs of 2, 1, and 0. If no one is elected and we proceed to the next round, the payoff is -1. If his fourth or fifth choice is elected, his payoff is -2 or -3.

Each preference pattern defines a new game.

Because every player has five options, there are $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 3125$ outcomes. We could represent them with payoffs on a 5-dimensional cube.

Let's instead look at an outcome and determine whether it is a Nash equilibrium in the two versions of the game. Assume A votes for A, B votes for B, C votes for C, D votes for C, and E votes for C. Then C is elected, and the payoffs for A, B, C, D, E are $-2, 1, 2, -1, 1$ in the early case game. Knight A is not happy but still has no reason to vote differently—if he voted for A or D instead, C would still be elected. But this outcome is not a Nash equilibrium, since D, knowing the voting pattern of the others, would rather have voted for B to obtain a tie and a payoff of 0.

In the exhausted case game, the payoffs for A, B, C, D, E for the same voting pattern are −2, 1, 2, 0, 1. Knight D still doesn't prefer C, but is now just content that somebody has been elected. That outcome is a Nash equilibrium in this version of the game. Nobody would, given the voting of the others, reconsider and vote differently. Knights A and D are still not happy, but they cannot unilaterally change this.

Let me show how we could search for Nash equilibria in the "exhausted knight" version. The idea is to start with any outcome, defined by a set of choices of the players. If all players are playing a best response to the other players' moves, we have a Nash equilibrium. Otherwise, at least one player does not play a best response yet—we let this player reconsider and play a best response. Then we evaluate the outcome again. Either we have a Nash equilibrium now, or we still have a player not playing the best response to the other players' moves. We continue, until we get a Nash equilibrium.

Look at the outcome where everybody votes for himself first. This would give a tie and everyone would prefer if his second choice would be elected. So, let's say D reconsiders and votes for B instead of for himself. Then B would be elected. B and E have better responses; A could vote for E instead for himself to get a tie and avoid the election of B. Now B, C, and E have better responses. Let's assume B plays his best response E to the other's moves. This voting pattern EECBE turns out to be a Nash equilibrium.

The process can be simulated in the `ExhaustedKnights` applet. Initially everybody votes his first preference. Change D's vote to B, then A's vote to E, then B's vote to E.

The process does not always terminate; confirm the following in the applet. We start with everyone voting for himself. Then A chooses a best response and votes for D. Then B chooses a best response and votes for C. After that D reconsiders and votes for B, then B reconsiders again, voting for himself again, and D reconsiders again, voting for himself again. After this we have an outcome that we had discussed earlier (voting pattern DBCDE) and the process could continue the same way forever.

Historical Remark In the Ph. D. thesis he wrote at Princeton University in 1950 the mathematician John Forbes Nash Jr. defined the equilibrium concept which is named after him. Later he did extraordinary work in other areas of mathematics. Around 1959 he became ill, suffering from paranoid schizophrenia throughout the 60s and the 70s. Surprisingly he recovered in the 80s. There is no Nobel prize in mathematics, but in 1994, with Reinhard Selten and John Harsanyi, Nash was awarded the Nobel prize in economics (to be more precise, the Nobel Memorial Prize in Economic Sciences). The award was given for his early work in game theory, including his definition of Nash equilibria and the existence theorem for them. The story of his life has been told in the book *A Beautiful Mind* by Sylvia Nasar [N1998], which in 2002 was made into an Oscar-winning movie with the same title.

2.3 Additional Topics

2.3.1 Best Response Digraphs

For a 2-player game, the best response information can be displayed in a graph. The bipartite **best response digraph** for two-player games is defined as follows: for every move of Ann we draw a white circle and for every move of Beth we draw a black circle. The circles are called the **vertices** of the digraph. From every white vertex we draw an arrow, an **arc**, towards black vertices that are best responses to the corresponding move of Ann. In the same way, arcs are drawn from black vertices towards best response white vertices. For Example 5, the best response digraph is shown in Figure 2.1.

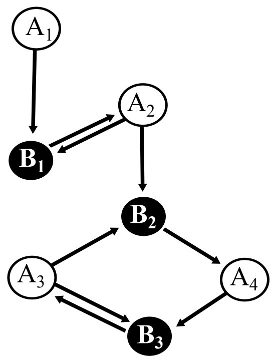

Figure 2.1. The best response digraph for Example 5

Condensed Best Response Digraphs for Symmetric Games

In symmetric games, like ADVERTISING and TWO BARS, it suffices to display a condensed version of the best response digraph. For every move (of either player— the game is symmetric, therefore both players have the same moves as options) we draw one vertex, and we draw an arc from move X to move Y if Beth's Y is a best response to Ann's X (and therefore also Ann's move Y is a best response to Beth's move X). See Figures 2.2 and 2.3 for the best response digraph and the condensed best response digraph for the TWO BARS example. We may have curved arcs in the condensed version (in our example from vertex 4 to itself) when one of Ann's move (in our example move 4) is the best response to the corresponding move (move 4) of Beth.

Figure 2.3. The condensed best response digraph for the symmetric TWO BARS game

For two-player games, Nash equilibria can be recognized from the best response digraph. Any pair of moves with arcs between them, one being the best response of the other, is a Nash equilibrium. For symmetric 2-person games, in the condensed best response digraph any pair of arcs between two vertices, or any loop starting at a vertex and pointing to itself represents a pure Nash equilibrium. Those stemming from loops are symmetric insofar as both players use the same move in them. In symmetric games they may seem more natural than asymmetric Nash equilibria.

2.3.2 2-Player Zero-sum Symmetric Games

For the class of games with these three attributes, Nash equilibria, if they exist, can be spotted easily, and the maximin point of view is the same as the Nash equilibrium view.

Theorem *In every symmetric zero-sum simultaneous game,*

1. *every pure Nash equilibrium has zero payoff for both players, and*
2. *every maximin move of Ann with security level 0 versus any maximin move of Beth with security level 0 forms a Nash equilibrium.*

Our first theorem! What is a theorem anyway? So far this chapter has contained mostly definitions, examples, and facts about examples, such as the fact that TWO BARS has one Nash equilibrium. **Theorems** are also facts, not about single concrete examples but about general abstract mathematical objects like simultaneous games.

We want to provide a **proof** for this theorem. Although proofs can be complicated, they just provide the reasons why the theorem is true. You can accept the truth of a theorem based on the authority of the author or teacher, so it is all right if you skip the (very few) proofs in this book on first reading. But if you want to understand a mathematical area, you also have to understand the reasoning behind the proofs, at least to some extent.

Proof

1. Look at an outcome where one player, say Ann, has a payoff of less than zero. Then the move chosen by Ann could not be her best response for the move chosen by Beth, since she can always get 0 by choosing the same move as Beth.
2. Ann's minimum payoff in each row cannot exceed 0, since, if both players choose the same option, both have a payoff of 0. Therefore the security level cannot be larger than 0.

   If Ann's minimum payoff in a row is less than 0, then each of Beth's best responses to the move of Ann corresponding to the row carries a payoff of more than 0 for Beth, therefore this move of Ann cannot be part of a Nash equilibrium by (1).

   Therefore, if the security level is less than 0, there are no pure Nash equilibria.

   If the security level (for both players, it is a symmetric game) equals 0, look at any maximin move for Ann and any maximin move for Beth. Then Ann's payoff in this move combination is at least 0, and Beth's payoff is at least 0. Since the game is zero-sum, both payoffs must be equal to 0. Then each move is the best response to the other move, and the move pair forms a pure Nash equilibrium.

It follows that a 2-player zero-sum symmetric game has no pure Nash equilibria provided the security level is less than 0. An example is ROCK-SCISSORS-PAPER.

One feature used in the analysis of simultaneous games is still missing. It is the topic of mixing moves and is discussed in Chapter 27.

Exercises

1. a) Write the matrices of the SIMULTANEOUS LEGISLATORS VOTE game in the variant where each of the three voters has also the option to abstain. The raise passes only if more agree than voting against. The loss of face by abstaining is relatively small, only $200.

   b) Solve the game, using the approaches discussed above.
2. Consider the following two-player game.

	L	M	R
U	1, 1	3, 4	2, 1
M	2, 4	2, 5	8, 1
D	3, 3	0, 4	0, 9

- Find the maximin moves for both players.
- Which moves are dominated?
- Find the bimatrix obtained by IESD.
- Find the bimatrix obtained by IEWD.

- Mark all best responses.
- Are there any Nash equilibria?

3. Analyze the following two-person zero-sum games for maximin moves, domination, best responses, and Nash equilibria:

a)

	L	R
U	1	2
D	3	4

b)

	L	R
U	1	2
D	4	3

c)

	L	R
U	1	3
D	2	4

d)

	L	R
U	1	3
D	4	2

e)

	L	R
U	1	4
D	2	3

f)

	L	R
U	1	4
D	3	2

4. Consider a two-person variant of the GUESS THE AVERAGE game: Ann and Beth simultaneously submit a number, 1, 2, 3, or 4. The player whose number is closest to 2/3 of the average of both numbers gets $1. Create the payoff bimatrix. Decide whether the game has a Nash equilibrium.
5. In the TWO BARS example a lack of tourists increases competition. Assume the number of natives is 4000. For which number of tourists would both bars choose $4 as the price for a beer? For which tourist numbers is $2 possible, and for which tourist numbers is $5 possible?
6. Write the payoff bimatrix of the following game. Find maximin moves, domination, best responses, and pure Nash equilibria.

   SCHEDULING A DINNER PARTY: Ann and Beth are not on speaking terms, but have a lot of common friends. Both want to invite them to a dinner party this weekend, either Friday or Saturday evening. Both slightly prefer Saturday. If both set the party at the same time, this will be considered a disaster with a payoff of -10 for both. If one plans the party on Friday and the other on Saturday, the one having the Saturday party gets a payoff of 5, and the other of 4.
7. Analyze the following game. Create payoff bimatrices consistent with the information given. Explain your choices. Then find the maximin moves, domination, and all pure Nash equilibria.

   SELECTING CLASS: Adam, Bill, and Cindy are registering for a foreign language class independently and simultaneously. The available classes are ITA100 and FRE100. They d not care much which, but they care with whom they share the class. Bill and Cindy want to be in the same class, but want to avoid Adam. Adam wants to be in the same class as Bill or Cindy, or even better, both.
8. DEADLOCK: Two players play a symmetric game where each can either cooperate or defect. If they cooperate, both get an payoff of 1. If they defect, both get a payoff of 2. If one cooperates but the other defects, the one cooperating gets a payoff of 0, and the one defecting a payoff of 3.

   Draw the bimatrix of the game. Find the maximin moves, possible domination, best responses, and find all pure Nash equilibria.
9. STAG HUNT: Two players play a symmetric game where each can hunt either stag or hare. If both hunt stag, both get an payoff of 3. If both hunt hare, both get a payoff of 1. If one hunts stag and the other hare, the stag hunter gets a payoff of 0, and the hare hunter a payoff of 2.

   Draw the bimatrix of the game. Find the maximin moves, possible domination, best responses, and find all pure Nash equilibria.
10. CHICKEN: Two players play a symmetric game where each can either play dove or hawk. If both play dove, both get an payoff of 2. If both play hawk, both get a payoff of 0. If one plays dove and the other hawk, the one playing dove gets a payoff of 1, and the other one a payoff of 3.

    Draw the bimatrix of the game. Find the maximin moves, possible domination, best responses, and find all pure Nash equilibria.

11. BULLY: Two players play the following game

	cooperate	defect
cooperate	2, 1	1, 3
defect	3, 0	0, 2

(compare [P1993]).
Find the maximin moves, possible domination, best responses, and find all pure Nash equilibria.

12. Two cars are meeting at an intersection and want to proceed as indicated by the arrows in Figure 2.4. Each player can proceed or move. If both proceed, there is an accident. A would have a payoff of −100 in this case, and B a payoff of −1000 (since B would be made responsible for the accident, since A has the right of way). If one yields and the other proceeds, the one yielding has a payoff of −5, and the other one of 5. If both yield, it takes a little longer until they can proceed, so both have a payoff of −10. Analyze this simultaneous game, draw the payoff bimatrix, and find pure Nash equilibria.

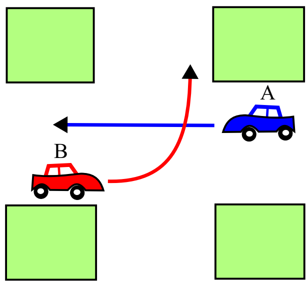

Figure 2.4. Two cars at a crossing

13. Three cars are meeting at an intersection and want to proceed as indicated by the arrows in Figure 2.5. Each player can proceed or move. If two with intersecting paths proceed, there is an accident. The one having the right of way has a payoff of −100 in this case, the other one a payoff of −1000. If a car proceeds without causing an accident, the payoff for that car is 5. If a car yields and all the others intersecting its path proceed, the yielding car has a payoff of −5. If a car yields and a conflicting path car as well, it takes a little longer until they can proceed, so both have a payoff of −10. Analyze this simultaneous game, draw the payoff bimatrices, and find all pure Nash equilibria.

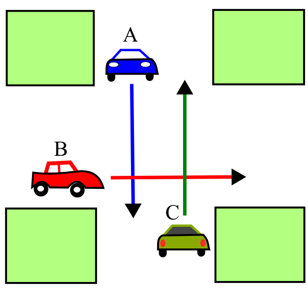

Figure 2.5. Three cars at a crossing

14. Solve the SIMULTANEOUS ULTIMATUM GAME. Display the payoff bimatrix, and investigate maximin moves, domination, best responses, and whether there are any equilibria.

15. Analyze a version of the BATTLE OF THE SEXES example where one partner has high love and the other low love. For the high love partner, being together with the partner is more important than being at the preferred location, whereas for the low love partner it is the opposite. Are there Nash equilibria?

16. Assume that a simultaneous two-player game has the best response digraph shown in Figure 2.6.

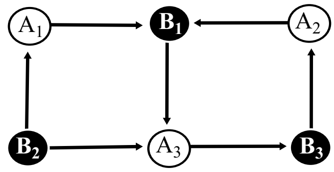

Figure 2.6. A best response digraph

Display a possible payoff bimatrix. Can you find a zero-sum payoff bimatrix generating this best response digraph?

17. In the 5 KNIGHTS game described in Example 8 with preferences as described there, determine whether the voting pattern ECEDE (A votes for E, B votes for C, etc.) forms a Nash equilibrium in the early case game or in the exhausted case game.
18. In the 5 KNIGHTS game described in Example 8 with preferences as described there, determine whether the voting pattern ACEBE (A votes for A, B votes for C, etc.) forms a Nash equilibrium in the early case game or in the exhausted case game.
19. Use the `5KNIGHTS` applet to find a Nash equilibrium for the early version of the 5 KNIGHTS game.

Project 1

Reacting fast or slow Assume the five players of the 5 KNIGHTS game are negotiating what to vote before actually voting. They start with the obvious proposal of everybody voting for his first choice. If somebody's move is not the best response to the other players' moves he changes the proposal, proposing his best response as his move. This is repeated until a Nash equilibrium is found.

The process is not unique, since if two or more players want to reconsider, only one will modify the proposal at a time. Discuss whether it is better in such a negotiation to always modify early, or to wait and see whether the others change the proposal first. Simulate the process for different preferences in the `5KNIGHTSRANDOM` applet with the assumption that A always reacts faster than B, B always reacts faster than C, and so on. Do this at least 30 times, and keep track how often each one of the players is elected in the resulting Nash equilibrium (which you hopefully get—there may also be cyclic cases as discussed in Example 8 of Chapter 27).

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