6

Combining Sequential and Simultaneous Moves

In Chapter 3, we considered games of purely sequential moves; Chapters 4 and 5 dealt with games of purely simultaneous moves. We developed concepts and techniques of analysis appropriate to the pure game types—trees and rollback equilibrium for sequential moves, payoff tables and Nash equilibrium for simultaneous moves. In reality, however, many strategic situations contain elements of both types of interaction. Also, although we used game trees (extensive forms) as the sole method of illustrating sequential-move games and game tables (strategic forms) as the sole method of illustrating simultaneous-move games, we can use either form for any type of game.

In this chapter, we examine many of these possibilities. We begin by showing how games that combine sequential and simultaneous moves can be solved by combining trees and payoff tables and by combining rollback and Nash equilibrium analysis in appropriate ways. Then we consider the effects of changing the nature of the interaction in a particular game. Specifically, we look at the effects of changing the rules of a game to convert sequential play into simultaneous play and vice versa and of changing the order of moves in sequential play. This topic gives us an opportunity to compare the equilibria found by using the concept of rollback, in a sequential-move game, with those found by using the Nash equilibrium concept, in the simultaneous version of the same game. From this comparison, we extend the concept of Nash equilibria to sequential-play games. It turns out that the rollback equilibrium is a special case, usually called a refinement, of these Nash equilibria.

1 GAMES WITH BOTH SIMULTANEOUS AND SEQUENTIAL MOVES

As mentioned several times thus far, most real games that you will encounter will be made up of numerous smaller components. Each of these components may entail simultaneous play or sequential play, so the full game requires you to be familiar with both. The most obvious examples of strategic interactions containing both sequential and simultaneous parts are those between two (or more) players over an extended period of time. You may play a number of different simultaneous-play games against your roommate during your year together: Your action in any one of these games is influenced by the history of your interactions up to then and by your expectations about the interactions to come. Also, many sporting events, interactions between competing firms in an industry, and political relationships are sequentially linked series of simultaneous-move games. Such games are analyzed by combining the tools presented in Chapter 3 (trees and rollback) and in Chapters 4 and 5 (payoff tables and Nash equilibria).[1] The only difference is that the actual analysis becomes more complicated as the number of moves and interactions increases.

A. Two-Stage Games and Subgames

Our main illustrative example for such situations includes two would-be telecom giants, CrossTalk and GlobalDialog. Each can choose whether to invest $10 billion in the purchase of a fiber-optic network. They make their investment decisions simultaneously. If neither chooses to make the investment, that is the end of the game. If one invests and the other does not, then the investor has to make a pricing decision for its telecom services. It can choose either a high price, which will attract 60 million customers, from each of whom it will make an operating profit of $400, or a low price, which will attract 80 million customers, from each of whom it will make an operating profit of $200. If both firms acquire fiber-optic networks and enter the market, then their pricing choices become a second simultaneous-move game. Each can choose either the high or the low price. If both choose the high price, they will split the total market equally; so each will get 30 million customers and an operating profit of $400 from each. If both choose the low price, again they will split the total market equally; so each will get 40 million customers and an operating profit of $200 from each. If one chooses the high price and the other the low price, then the low-price

[1] Sometimes the simultaneous part of the game will have equilibria in mixed strategies; then, the tools we develop in Chapter 7 will be required. We mention this possibility in this chapter where relevant and give you an opportunity to use such methods in exercises for the later chapters.

firm will get all the 80 million customers at that price, and the high-price firm will get nothing.

The interaction between CrossTalk and GlobalDialog forms a two-stage game. Of the four combinations of the simultaneous-move choices at the first (investment) stage, one ends the game, two lead to a second-stage (pricing) decision by just one player, and the fourth leads to a simultaneous-move (pricing) game at the second stage. We show this game pictorially in Figure 6.1.

Regarded as a whole, Figure 6.1 illustrates a game tree, but one that is more complex than the trees in Chapter 3. You can think of it as an elaborate "tree house" with multiple levels. The levels are shown in different parts of the same two-dimensional figure, as if you are looking down at the tree from a helicopter positioned directly above it.

The first-stage game is represented by the payoff table in the top-left quadrant of Figure 6.1. You can think of it as the first floor of the tree house. It has four "rooms." The room in the northwest corner corresponds to the "Don't invest" first-stage moves of both firms. If the firms' decisions take the game to this room, there are no further choices to be made, so we can think of it being like a terminal node of a tree in Chapter 3 and show the payoffs in the cell of the table;

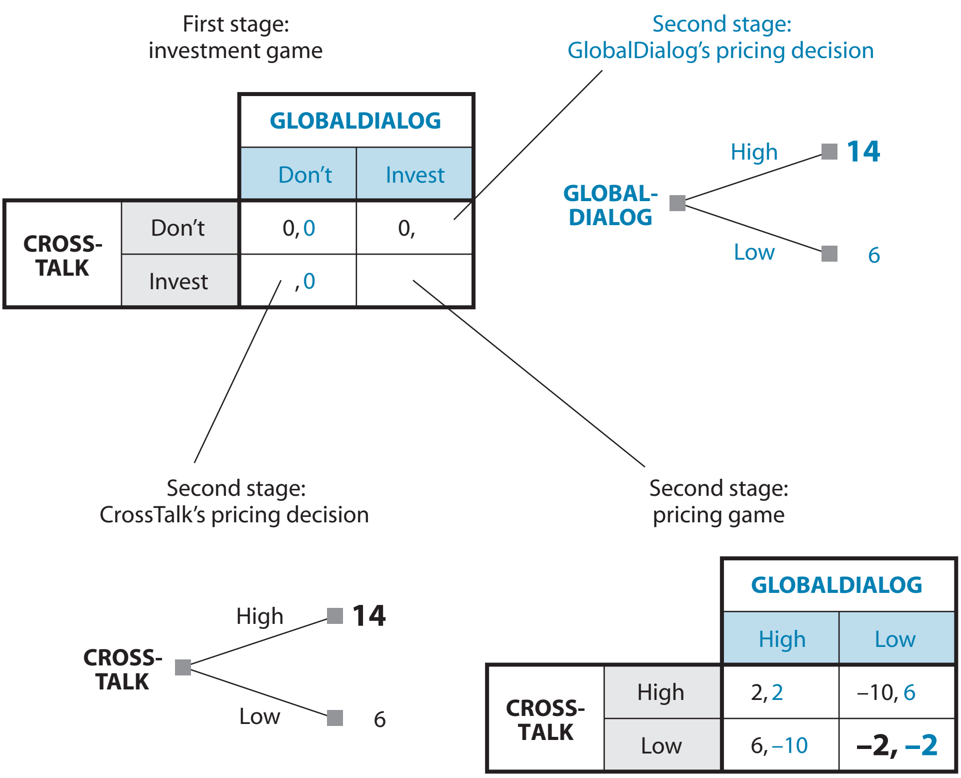


FIGURE 6.1 Two-Stage Game Combining Sequential and Simultaneous Moves

both firms get 0. However, all of the other combinations of actions for the two firms lead to rooms that lead to further choices; so we cannot yet show the payoffs in those cells. Instead, we show branches leading to the second floor. The northeast and southwest rooms show only the payoff to the firm that has not invested; the branches leading from each of these rooms take us to single-firm pricing decisions in the second stage. The southeast room leads to a multiroom second-floor structure within the tree house, which represents the second-stage pricing game that is played if both firms have invested in the first stage. This second-floor structure has four rooms corresponding to the four combinations of the two firms' pricing moves.

All of the second-floor branches and rooms are like terminal nodes of a game tree, so we can show the payoffs in each case. Payoffs here consist of each firm's operating profits minus the previous investment costs; payoff values are written in billions of dollars.

Consider the branch leading to the southwest corner of Figure 6.1. The game arrives in that corner if CrossTalk is the only firm that has invested. Then, if it chooses the high price, its operating profit is $400 × 60 million = $24 billion; after subtracting the $10 billion investment cost, its payoff is $14 billion, which we write as 14. In the same corner, if CrossTalk chooses the low price, then its operating profit is $200 × 80 million = $16 billion, yielding the payoff 6 after accounting for its original investment. In this situation, GlobalDialog's payoff is 0, as shown in the southwest room of the first floor of our tree. Similar calculations for the case in which GlobalDialog is the only firm to invest give us the payoffs shown in the northeast corner of Figure 6.1; again, the payoff of 0 for CrossTalk is shown in the northeast room of the first-stage game table.

If both firms invest, both play the second-stage pricing game illustrated in the southeast corner of the figure. When both choose the high price in the second stage, each gets operating profit of $400 × 30 million (half of the market), or $12 billion; after subtracting the $10 billion investment cost, each is left with a net profit of $2 billion, or a payoff of 2. If both firms choose the low price in the second stage, each gets operating profit of $200 × 40 million = $8 billion, and, after subtracting the $10 billion investment cost, each is left with a net loss of $2 billion, or a payoff of −2. Finally, if one firm charges the high price and the other firm the low price, then the low-price firm has operating profit of $200 × 80 million = $16 billion, leading to the payoff 6, while the high-price firm gets no operating profit and simply loses its $10 billion investment, for a payoff of −10.

As with any multistage game in Chapter 3, we must solve this game backward, starting with the second-stage game. In the two single-firm decision problems, we see at once that the high-price policy yields the higher payoff. We highlight this by showing that payoff in a larger-size type.

The second-stage pricing game has to be solved by using methods developed in Chapter 4. It is immediately evident, however, that this game is a prisoners'

		GLOBALDIALOG	
		Don't	Invest
CROSSTALK	Don't	0, 0	0, 14
	Invest	14, 0	−2, −2

FIGURE 6.2 First-Stage Investment Game (After Substituting Rolled-Back Payoffs from the Equilibrium of the Second Stage)

dilemma. Low is the dominant strategy for each firm; so the outcome is the room in the southeast corner of the second-stage game table; each firm gets payoff −2.[2] Again, we show these payoffs in a larger type size to highlight the fact that they are the payoffs obtained in the second-stage equilibrium.

Rollback now tells us that each first-stage configuration of moves should be evaluated by looking ahead to the equilibrium of the second-stage game (or the optimum second-stage decision) and the resulting payoffs. We can therefore substitute the payoffs that we have just calculated into the previously empty or partly empty rooms on the first floor of our tree house. This substitution gives us a first floor with known payoffs, shown in Figure 6.2.

Now we can use the methods of Chapter 4 to solve this simultaneous-move game. You should immediately recognize the game in Figure 6.2 as a chicken game. It has two Nash equilibria, each of which entails one firm choosing Invest and the other choosing Don't. The firm that invests makes a huge profit; so each firm prefers the equilibrium in which it is the investor while the other firm stays out. In Chapter 4, we briefly discussed the ways in which one of the two equilibria might get selected. We also pointed out the possibility that each firm might try to get its preferred outcome, with the result that both of them invest and both lose money. Indeed, this is what seems to have happened in the real-life play of this game. In Chapter 7, we investigate this type of game further, showing that it has a third Nash equilibrium, in mixed strategies.

Analysis of Figure 6.2 shows that the first-stage game in our example does not have a unique Nash equilibrium. This problem is not too serious, because we can leave the solution ambiguous to the extent that was done in the preceding paragraph. Matters would be worse if the second-stage game did not have a unique equilibrium. Then it would be essential to specify the precise process by which an outcome gets selected so that we could figure out the second-stage payoffs and use them to roll back to the first stage.

[2] As is usual in a prisoners' dilemma, if the firms could successfully collude and charge high prices, both could get the higher payoff of 2. But this outcome is not an equilibrium because each firm is tempted to cheat to try to get the much higher payoff of 6.

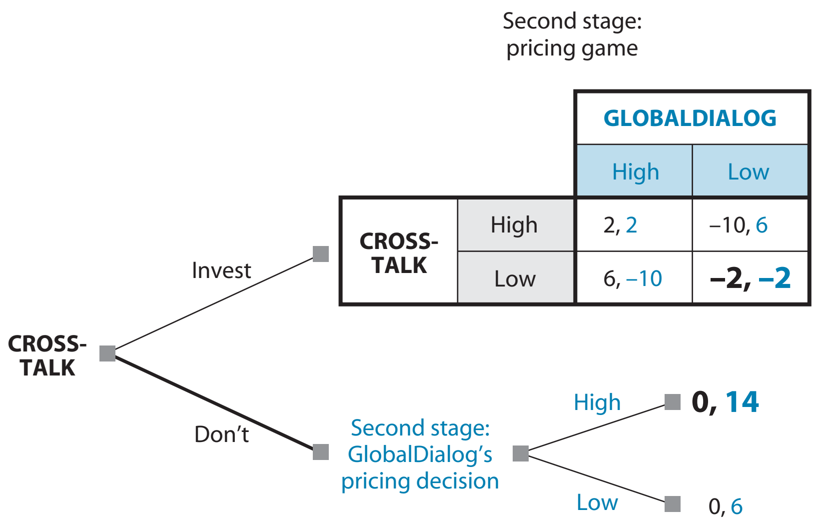
The second-stage pricing game shown in the table in the bottom-right quadrant of Figure 6.1 is one part of the complete two-stage game. However, it is also a full-fledged game in its own right, with a fully specified structure of players, strategies, and payoffs. To bring out this dual nature more explicitly, it is called a **subgame** of the full game.

More generally, a subgame is the part of a multimove game that begins at a particular node of the original game. The tree for a subgame is then just that part of the tree for the full game that takes this node as its root, or initial, node. A multimove game has as many subgames as it has decision nodes.

B. Configurations of Multistage Games

In the multilevel game illustrated in Figure 6.1, each stage consists of a simultaneous-move game. However, that may not always be the case. Simultaneous and sequential components may be mixed and matched in any way. We give two more examples to clarify this point and to reinforce the ideas introduced in the preceding section.

The first example is a slight variation of the CrossTalk–GlobalDialog game. Suppose one of the firms—say, GlobalDialog—has already made the $10 billion investment in the fiber-optic network. CrossTalk knows of this investment and now has to decide whether to make its own investment. If CrossTalk does not invest, then GlobalDialog will have a simple pricing decision to make. If CrossTalk invests, then the two firms will play the second-stage pricing game already described. The tree for this multistage game has conventional branches at the initial node and has a simultaneous-move subgame starting at one of the nodes to which these initial branches lead. The complete tree is shown in Figure 6.3.



		GLOBALDIALOG	
		High	Low
CROSS-TALK	High	2, 2	−10, 6
	Low	6, −10	**−2, −2**

FIGURE 6.3 Two-Stage Game When One Firm Has Already Invested

When the tree has been set up, it is easy to analyze the game. We show the rollback analysis in Figure 6.3 by using large type for the equilibrium payoffs that result from the second-stage game or decision and a thicker branch for CrossTalk's first-stage choice. In words, CrossTalk figures out that, if it invests, the ensuing prisoners' dilemma of pricing will leave it with payoff −2, whereas staying out will get it 0. Thus, it prefers the latter. GlobalDialog gets 14 instead of the −2 that it would have gotten if CrossTalk had invested, but CrossTalk's concern is to maximize its own payoff and not to ruin GlobalDialog deliberately.

This analysis does raise the possibility, though, that GlobalDialog may try to get its investment done quickly before CrossTalk makes its decision so as to ensure its most preferred outcome from the full game. And CrossTalk may try to beat GlobalDialog to the punch in the same way. In Chapter 9, we study some methods, called strategic moves, that may enable players to secure such advantages.

Our second example comes from football. Before each play, the coach for the offense chooses the play that his team will run; simultaneously, the coach for the defense sends his team out with instructions on how they should align themselves to counter the offense. Thus, these moves are simultaneous. Suppose the offense has just two alternatives, a safe play and a risky play, and the defense may align itself to counter either of them. If the offense has planned to run the risky play and the quarterback sees the defensive alignment that will counter it, he can change the play at the line of scrimmage. And the defense, hearing the change, can respond by changing its own alignment. Thus, we have a simultaneous-move game at the first stage, and one of the combination of choices of moves at this stage leads to a sequential-move subgame. Figure 6.4 shows the complete tree.

This is a zero-sum game in which the offense's payoffs are measured in the number of yards that it expects to gain, and the defense's payoffs are exactly the opposite, measured in the number of yards it expects to give up. The safe play for the offense gets it 2 yards, even if the defense is ready for it; if the defense is not ready for it, the safe play does not do much better, gaining 6 yards. The risky play, if it catches the defense unready to cover it, gains 30 yards. But if the defense is ready for the risky play, the offense loses 10 yards. We show this set of payoffs of −10 for the offense and 10 for the defense at the terminal node where the offense does not change the play. If the offense changes the play (back to safe), the payoffs are (2, −2) if the defense responds and (6, −6) if it does not; these payoffs are the same as those that arise when the offense plans the safe play from the start.

We show the chosen branches in the sequential subgame as thick lines in Figure 6.4. It is easy to see that, if the offense changes its play, the defense will respond to keep its payoff at −2 rather than −6 and that the offense should change the play to get 2 rather than −10. Rolling back, we should put the resulting set of

First stage:
coaches choose alignment

		DEFENSE TO COVER	
		Safe	Risky
OFFENSE TO PLAY	Safe	2, −2	6, −6
	Risky	30, −30	

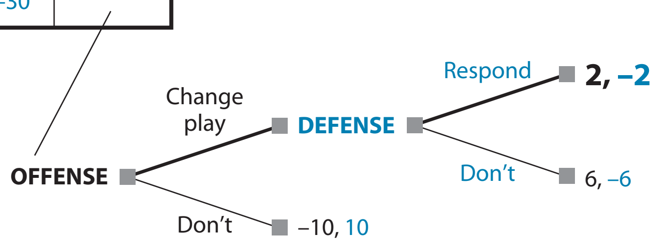


FIGURE 6.4 Simultaneous-Move First Stage Followed by Sequential Moves

payoffs, (2, −2), in the bottom-right cell of the simultaneous-move game of the first stage. Then we see that this game has no Nash equilibrium in pure strategies. The reason is the same as that in the tennis game of Chapter 4, Section 7; one player (defense) wants to match the moves (align to counter the play that the offense is choosing) while the other (offense) wants to unmatch the moves (catch the defense in the wrong alignment). In Chapter 7, we show how to calculate the mixed-strategy equilibrium of such a game. It turns out that the offense should choose the risky play with probability 1/8, or 12.5%.

2 CHANGING THE ORDER OF MOVES IN A GAME

The games considered in preceding chapters were presented as either sequential or simultaneous in nature. We used the appropriate tools of analysis to predict equilibria in each type of game. In Section 1 of this chapter, we discussed games with elements of both sequential and simultaneous play. These games required both sets of tools to find solutions. But what about games that could be played either sequentially or simultaneously? How would changing the play of a particular game and thus changing the appropriate tools of analysis alter the expected outcomes?

The task of turning a sequential-play game into a simultaneous one requires changing only the timing or observability with which players make their choices of moves. Sequential-move games become simultaneous if the players cannot observe moves made by their rivals before making their own choices. In that case, we would analyze the game by searching for a Nash equilibrium rather than for a rollback equilibrium. Conversely, a simultaneous-move game could become sequential if one player were able to observe the other's move before choosing her own.

Any changes to the rules of the game can also change its outcomes. Here, we illustrate a variety of possibilities that arise owing to changes in different types of games.

A. Changing Simultaneous-Move Games into Sequential-Move Games

I. FIRST-MOVER ADVANTAGE A first-mover advantage may emerge when the rules of a game are changed from simultaneous to sequential play. At a minimum, if the simultaneous-move version has multiple equilibria, the sequential-move version enables the first mover to choose his preferred outcome. We illustrate such a situation with the use of chicken, the game in which two teenagers drive toward each other in their cars, both determined not to swerve. We reproduce the strategic form of Figure 4.14 from Chapter 4 in Figure 6.5a and two extensive forms, one for each possible ordering of play, in Figure 6.5b and c.

Under simultaneous play, the two outcomes in which one player swerves (is "chicken") and the other goes straight (is "tough") are both pure-strategy Nash equilibria. Without specification of some historical, cultural, or other convention, neither has a claim to be a focal point. Our analysis in Chapter 4 suggested that coordinated play could help the players in this game, perhaps through an agreement to alternate between the two equilibria.

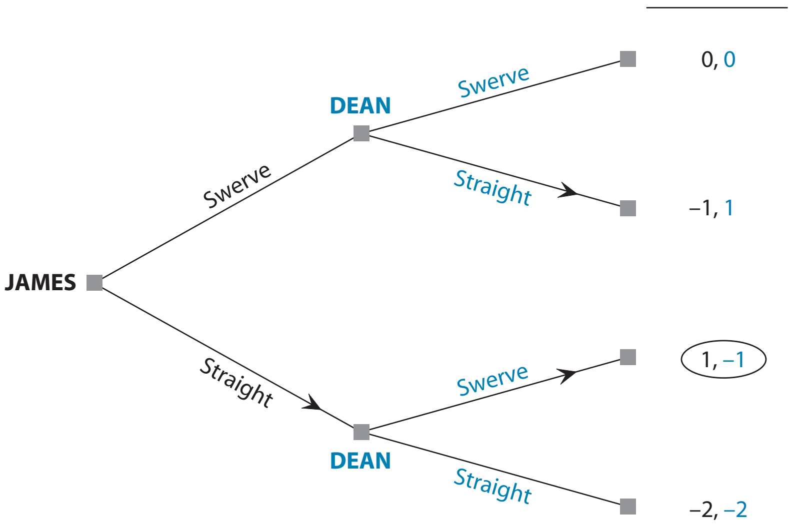
When we alter the rules of the game to allow one of the players the opportunity to move first, there are no longer two equilibria. Rather, we see that the second mover's equilibrium strategy is to choose the action opposite that chosen by the first mover. Rollback then shows that the first mover's equilibrium strategy is Straight. We see in Figure 6.5b and c that allowing one person to move first and to be observed making the move results in a single rollback equilibrium in which the first mover gets a payoff of 1, while the second mover gets a payoff of -1. The actual play of the game becomes almost irrelevant under such rules, which may make the sequential version uninteresting to many observers. Although teenagers might not want to play such a game with the rule change, the strategic consequences of the change are significant.

II. SECOND-MOVER ADVANTAGE In other games, a second-mover advantage may emerge when simultaneous play is changed into sequential play. This result can be illustrated using the tennis game of Chapter 4. Recall that, in that game, Evert is planning the location of her return while Navratilova considers where to cover. The version considered earlier assumed that both players were skilled at disguising their intended moves until the very last moment so that they moved at

(a) Simultaneous play

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

(b) Sequential play: James moves first



(c) Sequential play: Dean moves first

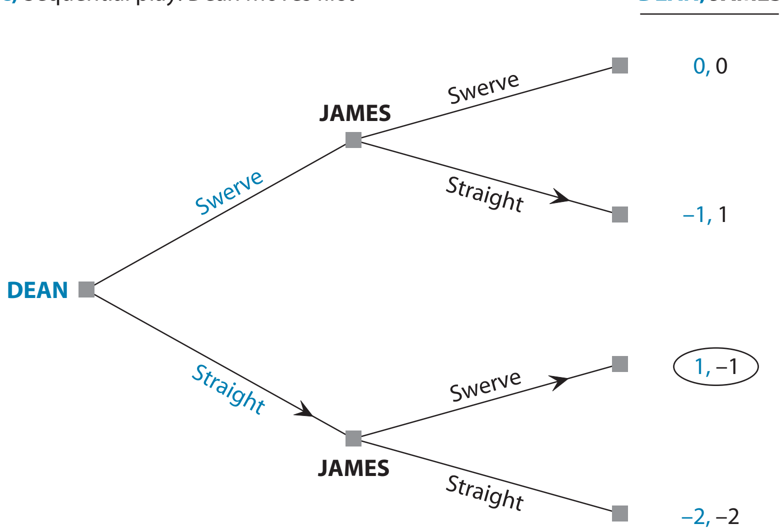


FIGURE 6.5 Chicken in Simultaneous-Play and Sequential-Play Versions

essentially the same time. If Evert's movement as she goes to hit the ball belies her shot intentions, however, then Navratilova can react and move second in the game. In the same way, if Navratilova leans toward the side that she intends to cover before Evert actually hits her return, then Evert is the second mover.

The simultaneous-play version of this game has no equilibrium in pure strategies. In each ordering of the sequential version, however, there is a unique rollback equilibrium outcome; the equilibrium differs, depending on who moves first. If Evert moves first, then Navratilova chooses to cover whichever direction Evert chooses and Evert opts for a down-the-line shot. Each player is expected to win the point half the time in this equilibrium. If the order is reversed, Evert chooses to send her shot in the opposite direction from that which Navratilova covers; so Navratilova should move to cover crosscourt. In this case, Evert is expected to win the point 80% of the time. The second mover does better by being able to respond optimally to the opponent's move. You should be able to draw game trees similar to those in Figure 6.5b and c that illustrate exactly these outcomes.

We return to the simultaneous version of this game in Chapter 7. There we show that it does have a Nash equilibrium in mixed strategies. In that equilibrium, Evert succeeds on average 62% of the time. Her success rate in the mixed-strategy equilibrium of the simultaneous game is thus better than the 50% that she gets by moving first but is worse than the 80% that she gets by moving second in the two sequential-move versions.

III. BOTH PLAYERS MAY DO BETTER That a game may have a first-mover or a second-mover advantage, which is suppressed when moves have to be simultaneous but emerges when an order of moves is imposed, is quite intuitive. Somewhat more surprising is the possibility that both players may do better under one set of rules of play than under another. We illustrate this possibility by using the game of monetary and fiscal policies played by the Federal Reserve and Congress. In Chapter 4, we studied this game with simultaneous moves; we reproduce the payoff table (Figure 4.5) as Figure 6.6a and show the two sequential-move versions as Figure 6.6b and c. For brevity, we write the strategies as Balance and Deficit instead of Budget Balance and Budget Deficit for Congress and as High and Low instead of High Interest Rates and Low Interest Rates for the Fed.

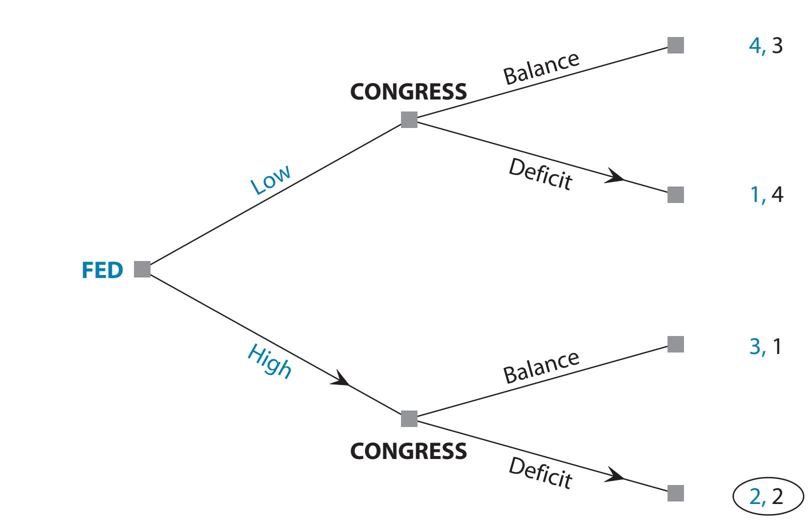
In the simultaneous-move version, Congress has a dominant strategy (Deficit), and the Fed, knowing this, chooses High, yielding payoffs of 2 to both players. Almost the same thing happens in the sequential version where the Fed moves first. The Fed foresees that, for each choice it might make, Congress will respond with Deficit. Then High is the better choice for the Fed, yielding 2 instead of 1.

(a) Simultaneous moves

		FEDERAL RESERVE	
		Low interest rates	High interest rates
CONGRESS	Budget balance	3, 4	1, 3
	Budget deficit	4, 1	2, 2

(b) Sequential moves: Fed moves first

FED, CONGRESS



(c) Sequential moves: Congress moves first

CONGRESS, FED

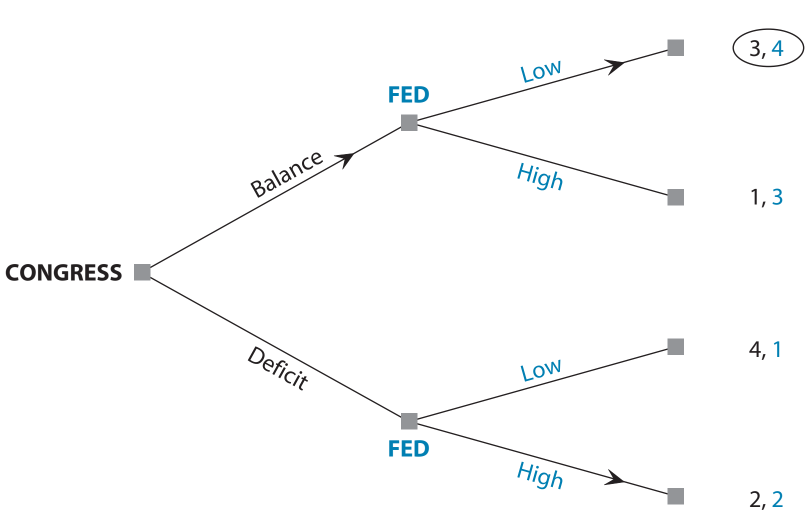


FIGURE 6.6 Three Versions of the Monetary–Fiscal Policy Game

But the sequential-move version where Congress moves first is different. Now Congress foresees that, if it chooses Deficit, the Fed will respond with High, whereas, if it chooses Balance, the Fed will respond with Low. Of these two developments, Congress prefers the latter, where it gets payoff 3 instead of 2. Therefore, the rollback equilibrium with this order of moves is for Congress to choose a balanced budget and the Fed to respond with low interest rates. The resulting payoffs, 3 for Congress and 4 for the Fed, are better for both players than those of the other two versions.

The difference between the two outcomes is even more surprising because the better outcome obtained in Figure 6.6c results from Congress choosing Balance, which is its dominated strategy in Figure 6.6a. To resolve the apparent paradox, one must understand more precisely the meaning of dominance. For Deficit to be a dominant strategy, it must be better than Balance from Congress's perspective for *each given* choice of the Fed. This type of comparison between Deficit and Balance is relevant in the simultaneous-move game because there Congress must make a decision without knowing the Fed's choice. Congress must think through, or formulate a belief about, the Fed's action and choose its best response to that. In our example, this best response is always Deficit for Congress. The concept of dominance is also relevant with sequential moves if Congress moves second, because then it knows what the Fed has already done and merely picks its best response, which is always Deficit. However, if Congress moves first, it cannot take the Fed's choice as *given*. Instead, it must recognize how the Fed's second move will be affected by its own first move. Here it knows that the Fed will respond to Deficit with High and to Balance with Low. Congress is then left to choose between these two alternatives; its most preferred outcome of Deficit and Low becomes irrelevant because it is precluded by the Fed's response.

The idea that dominance may cease to be a relevant concept for the first mover reemerges in Chapter 9. There we consider the possibility that one player or the other may deliberately change the rules of a game to become the first mover. Players can alter the outcome of the game in their favor in this way.

Suppose that the two players in our current example could choose the order of moves in the game. In this case, they would agree that Congress should move first. Indeed, when budget deficits and inflation threaten, the chairs of the Federal Reserve in testimony before various congressional committees often offer such deals; they promise to respond to fiscal discipline by lowering interest rates. But it is often not enough to make a verbal deal with the other player. The technical requirements of a first move—namely, that it be observable to the second mover and not reversible thereafter—must be satisfied. In the context of macroeconomic policies, it is fortunate that the legislative process of fiscal policy in the United States is both very visible and very slow, whereas monetary policy can be changed quite quickly in a meeting of the Federal Reserve Board. Therefore, the sequential play where Congress moves first and the Fed moves second is quite realistic.

IV. NO CHANGE IN OUTCOME So far, we have encountered only games that yield different outcomes when played sequentially instead of simultaneously. But certain games have the same outcomes in both types of play and regardless of the order of moves. This result generally arises only when both or all players have dominant strategies. We show that it holds for the prisoners' dilemma.

Consider the prisoners' dilemma game of Chapter 4, in which a husband and wife are being questioned regarding their roles in a crime. The Nash equilibrium of that simultaneous-play game is for each player to confess (or to defect from cooperating with the other). But how would play transpire if one spouse made an observable choice before the other chose at all? Using rollback on a game tree similar to that in Figure 6.5b (which you can draw on your own as a check of our analysis) would show that the second player does best to confess if the first has already confessed (10 years rather than 25 years in jail), and the second player also does best to confess if the first has denied (1 year rather than 3 years in jail). Given these choices by the second player, the first player does best to confess (10 years rather than 25 years in jail). The equilibrium entails 10 years of jail for both spouses regardless of which one moves first. Thus, the equilibrium is the same in all three versions of this game!

B. Other Changes in the Order of Moves

The preceding section presented various examples in which the rules of the game were changed from simultaneous play to sequential play. We saw how and why such rule changes can change the outcome of a game. The same examples also serve to show what happens if the rules are changed in the opposite direction, from sequential to simultaneous moves. Thus, if a first- or a second-mover advantage exists with sequential play, it can be lost under simultaneous play. And if a specific order benefits both players, then losing the order can hurt both.

The same examples also show us what happens if the rules are changed to reverse the order of play while keeping the sequential nature of a game unchanged. If there is a first-mover or a second-mover advantage, then the player who shifts from moving first to moving second may benefit or lose accordingly, with the opposite change for the other player. And if one order is in the common interests of both, then an externally imposed change of order can benefit or hurt them both.

3 CHANGE IN THE METHOD OF ANALYSIS

Game trees are the natural way to display sequential-move games, and payoff tables are the natural representation of simultaneous-move games. However, each technique can be adapted to the other type of game. Here we show how to translate the information contained in one illustration to an illustration of the other type. In the process, we develop some new ideas that will prove useful in subsequent analysis of games.

A. Illustrating Simultaneous-Move Games by Using Trees

Consider the game of the passing shot in tennis as originally described in Chapter 4, where the action is so quick that moves are truly simultaneous, as shown in Figure 6.5a. But suppose we want to show the game in extensive form—that is, by using a tree rather than in a table as in Figure 4.14. We show how this can be done in Figure 6.7.

To draw the tree in the figure, we must choose one player—say, Evert—to make her choice at the initial node of the tree. The branches for her two choices, DL and CC, then end in two nodes, at each of which Navratilova makes her choices. However, because the moves are actually simultaneous, Navratilova must choose without knowing what Evert has picked. That is, she must choose without knowing whether she is at the node following Evert's DL branch or the one following Evert's CC branch. Our tree diagram must in some way show this lack of information on Navratilova's part.

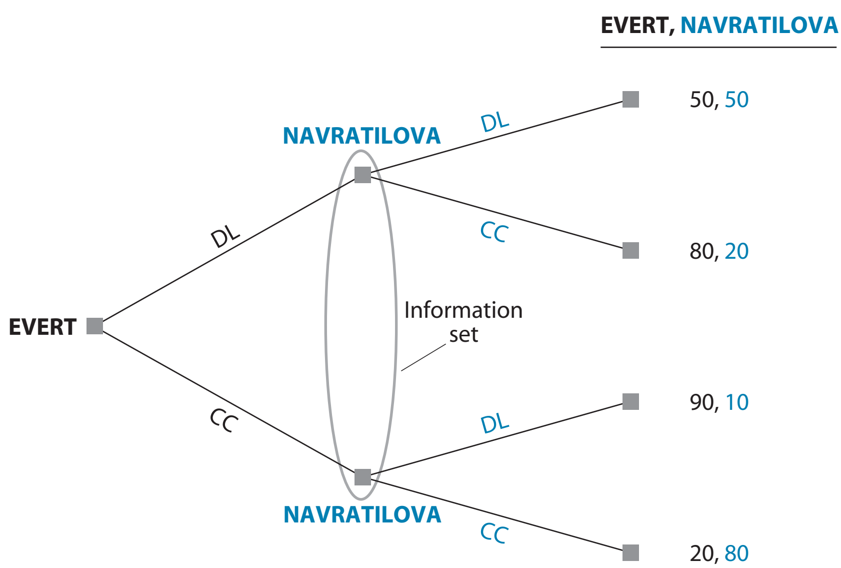


FIGURE 6.7 Simultaneous-Move Tennis Game Shown in Extensive Form

We illustrate Navratilova's strategic uncertainty about the node from which her decision is being made by drawing an oval to surround the two relevant nodes. (An alternative is to connect them by a dotted line; a dotted line is used to distinguish it from the solid lines that represent the branches of the tree.) The nodes within this oval or balloon are called an **information set** for the player who moves there. Such a set indicates the presence of imperfect information for the player; she cannot distinguish between the nodes in the set, given her available information (because she cannot observe the row player's move before making her own). As such, her strategy choice from within a single information set must specify the same move at all the nodes contained in it. That is, Navratilova must choose either DL at both the nodes in this information set or CC at both of them. She cannot choose DL at one and CC at the other, as she could in Figure 6.5b, where the game had sequential moves and she moved second.

Accordingly, we must adapt our definition of strategy. In Chapter 3, we defined a strategy as a complete plan of action, specifying the move that a player would make at each *node* where the rules of the game specified that it was her turn to move. We should now more accurately redefine a strategy as a complete plan of action, specifying the move that a player would make at each *information set* at whose nodes the rules of the game specify that it is her turn to move.

The concept of an information set is also relevant when a player faces external uncertainty about some conditions that affect his decision, rather than about another player's moves. For example, a farmer planting a crop is uncertain about the weather during the growing season, although he knows the probabilities of various alternative possibilities from past experience or meteorological forecasts. We can regard the weather as a random choice of an outside player, Nature, who has no payoffs but merely chooses according to known probabilities.[3] We can then enclose the various nodes corresponding to Nature's moves into an information set for the farmer, constraining the farmer's choice to be the same at all of these nodes. Figure 6.8 illustrates this situation.

Using the concept of an information set, we can formalize the concepts of perfect and imperfect information in a game, which we introduced in Chapter 2 (Section 2.D). A game has perfect information if it has neither strategic nor external uncertainty, which will happen if it has no information sets enclosing two or more nodes. Thus, a game has perfect information if all of its information sets consist of singleton nodes.

Although this representation is conceptually simple, it does not provide any simpler way of solving the game. Therefore, we use it only occasionally, where

[3] Some people believe that Nature is actually a malevolent player who plays a zero-sum game with us, so its payoffs are higher when ours are lower. For example, it is more likely to rain if we have forgotten to bring an umbrella. We understand such thinking, but it does not have real statistical support.

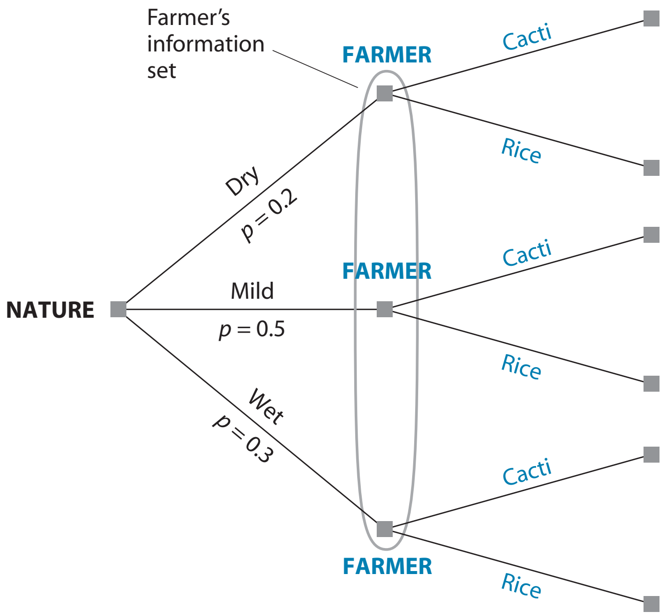


FIGURE 6.8 Nature and Information Sets

it conveys some point more simply. Some examples of game illustrations using information sets can be found later in Chapters 8 and 14.

B. Showing and Analyzing Sequential-Move Games in Strategic Form

Consider now the sequential-move game of monetary and fiscal policy from Figure 6.6c, in which Congress has the first move. Suppose we want to show it in normal or strategic form—that is, by using a payoff table. The rows and the columns of the table are the strategies of the two players. We must therefore begin by specifying the strategies.

For Congress, the first mover, listing its strategies is easy. There are just two moves—Balance and Deficit—and they are also the two strategies. For the second mover, matters are more complex. Remember that a strategy is a complete plan of action, specifying the moves to be made at each node where it is a player's turn to move. Because the Fed gets to move at two nodes (and because we are supposing that this game actually has sequential moves and so the two nodes are not confounded into one information set) and can choose either Low or High at each node, there are four combinations of its choice patterns. These combinations are (1) Low if Balance, High if Deficit (we write this as "L if B, H if D" for short); (2) High if Balance, Low if Deficit ("H if B, L if D" for short); (3) Low always; and (4) High always.

We show the resulting two-by-four payoff matrix in Figure 6.9. The last two columns are no different from those for the two-by-two payoff matrix for the

		FED			
		L if B, H if D	H if B, L if D	Low always	High always
CONGRESS	Balance	3, 4	1, 3	3, 4	1, 3
	Deficit	2, 2	4, 1	4, 1	2, 2

FIGURE 6.9 Sequential-Move Game of Monetary and Fiscal Policy in Strategic Form

game under simultaneous-move rules (Figure 6.6a). This is because, if the Fed is choosing a strategy in which it makes the same move always, it is just as if the Fed were moving without taking into account what Congress had done; it is as if their moves were simultaneous. But calculation of the payoffs for the first two columns, where the Fed's second move does depend on Congress's first move, needs some care.

To illustrate, consider the cell in the first row and the second column. Here Congress is choosing Balance, and the Fed is choosing "H if B, L if D." Given Congress's choice, the Fed's actual choice under this strategy is High. Then the payoffs are those for the Balance and High combination—namely, 1 for Congress and 3 for the Fed.

Best-response analysis quickly shows that the game has two pure-strategy Nash equilibria, which we show by shading the cells gray. One is in the top-left cell, where Congress's strategy is Balance and the Fed's is "L if B, H if D," and so the Fed's actual choice is L. This outcome is just the rollback equilibrium of the sequential-move game. But there is another Nash equilibrium in the bottom-right cell, where Congress chooses Deficit and the Fed chooses "High always." As always in a Nash equilibrium, neither player has a clear reason to deviate from the strategies that lead to this outcome. Congress would do worse by switching to Balance, and the Fed could do no better by switching to any of its other three strategies, although it could do just as well with "L if B, H if D."

The sequential-move game, when analyzed in its extensive form, produced just one rollback equilibrium. But when analyzed in its normal or strategic form, it has two Nash equilibria. What is going on?

The answer lies in the different nature of the logic of Nash and rollback analyses. Nash equilibrium requires that neither player have a reason to deviate, given the strategy of the other player. However, rollback does not take the strategies of later movers as given. Instead, it asks what would be optimal to do if the opportunity to move actually arises.

In our example, the Fed's strategy of "High always" does not satisfy the criterion of being optimal if the opportunity to move actually arises. If Congress chose Deficit, then High is indeed the Fed's optimal response. However, if Congress chose Balance and the Fed had to respond, it would want to choose

Low, not High. So “High always” does not describe the Fed’s optimal response in all possible configurations of play and cannot be a rollback equilibrium. But the logic of Nash equilibrium does not impose such a test, instead regarding the Fed’s “High always” as a strategy that Congress could legitimately take as given. If it does so, then Deficit is Congress’s best response. And, conversely, “High always” is one best response of the Fed to Congress’s Deficit (although it is tied with “L if B, H if D”). Thus, the pair of strategies “Deficit” and “High always” are mutual best responses and constitute a Nash equilibrium, although they do not constitute a rollback equilibrium.

We can therefore think of rollback as a further test, supplementing the requirements of a Nash equilibrium and helping to select from among multiple Nash equilibria of the strategic form. In other words, it is a refinement of the Nash equilibrium concept.

To state this idea somewhat more precisely, recall the concept of a subgame. At any one node of the full game tree, we can think of the part of the game that begins there as a subgame. In fact, as successive players make their choices, the play of the game moves along a succession of nodes, and each move can be thought of as starting a subgame. The equilibrium derived by using rollback corresponds to one particular succession of choices in each subgame and gives rise to one particular path of play. Certainly, other paths of play are consistent with the rules of the game. We call these other paths **off-equilibrium paths**, and we call any subgames that arise along these paths **off-equilibrium subgames**, for short.

With this terminology, we can now say that the equilibrium path of play is itself determined by the players’ expectations of what would happen if they chose a different action—if they moved the game to an off-equilibrium path and started an off-equilibrium subgame. Rollback requires that all players make their best choices in *every* subgame of the larger game, whether or not the subgame lies along the path to the ultimate equilibrium outcome.

Strategies are complete plans of action. Thus, a player’s strategy must specify what she will do in each eventuality, or each and every node of the game, whether on or off the equilibrium path, where it is her turn to act. When one such node arrives, only the plan of action starting there—namely, the part of the full strategy that pertains to the subgame starting at that node—is pertinent. This part is called the **continuation** of the strategy for that subgame. Rollback requires that the equilibrium strategy be such that its continuation in every subgame is optimal for the player whose turn it is to act at that node, whether or not the node and the subgame lie on the equilibrium path of play.

Return to the monetary policy game with Congress moving first, and consider the second Nash equilibrium that arises in its strategic form. Here the path of play is for Congress to choose Deficit and the Fed to choose High.

On the equilibrium path, High is indeed the Fed's best response to Deficit. Congress's choice of Balance would be the start of an off-equilibrium path. It leads to a node where a rather trivial subgame starts—namely, a decision by the Fed. The Fed's purported equilibrium strategy "High always" asks it to choose High in this subgame. But that is not optimal; this second equilibrium is specifying a nonoptimal choice for an off-equilibrium subgame.

In contrast, the equilibrium path of play for the Nash equilibrium in the upper-left corner of Figure 6.9 is for Congress to choose Balance and the Fed to follow with Low. The Fed is responding optimally on the equilibrium path. The off-equilibrium path would have Congress choosing Deficit, and the Fed, given its strategy of "L if B, H if D," would follow with High. It is optimal for the Fed to respond to Deficit with High, so the strategy remains optimal off the equilibrium path, too.

The requirement that continuation of a strategy remain optimal under all circumstances is important because the equilibrium path itself is the result of players' thinking strategically about what would happen if they did something different. A later player may try to achieve an outcome that she would prefer by threatening the first mover that certain actions would be met with dire responses or by promising that certain other actions would be met with nice responses. But the first mover will be skeptical of the **credibility** of such threats and promises. The only way to remove that doubt is to check if the stated responses would actually be optimal if the need arose. If the responses are not optimal, then the threats or promises are not credible, and the responses would not be observed along the equilibrium path of play.

The equilibrium found by using rollback is called a **subgame-perfect equilibrium (SPE)**. It is a set of strategies (complete plans of action), one for each player, such that, at every node of the game tree, whether or not the node lies along the equilibrium path of play, the continuation of the same strategy in the subgame starting at that node is optimal for the player who takes the action there. More simply, an SPE requires players to use strategies that constitute a Nash equilibrium in every subgame of the larger game.

In fact, as a rule, in games with finite trees and perfect information, where players can observe every previous action taken by all other players so that there are no multiple nodes enclosed in one information set, rollback finds the unique (except for trivial and exceptional cases of ties) subgame-perfect equilibrium of the game. Consider: If you look at any subgame that begins at the last decision node for the last player who moves, the best choice for that player is the one that gives her the highest payoff. But that is precisely the action chosen with the use of rollback. As players move backward through the game tree, rollback eliminates all unreasonable strategies, including incredible threats or promises, so that the collection of actions ultimately selected is the SPE. Therefore, for the purposes of this book, subgame perfectness is just a fancy name for rollback. At more advanced levels of game theory, where games include complex information structures and information sets, subgame perfectness becomes a richer notion.

4 THREE-PLAYER GAMES

We have restricted the discussion so far in this chapter to games with two players and two moves each. But the same methods also work for some larger and more general examples. We now illustrate this by using the street–garden game of Chapter 3. Specifically, we (1) change the rules of the game from sequential to simultaneous moves and then (2) keep the moves sequential but show and analyze the game in its strategic form. First we reproduce the tree of that sequential-move game (Figure 3.6) as Figure 6.10 here and remind you of the rollback equilibrium.

The equilibrium strategy of the first mover (Emily) is simply a move, "Don't contribute." The second mover chooses from among four possible strategies (choice of two responses at each of two nodes) and chooses the strategy "Don't contribute (D) if Emily has chosen her Contribute, and Contribute (C) if Emily has chosen her Don't contribute," or, more simply, "D if C, C if D," or even

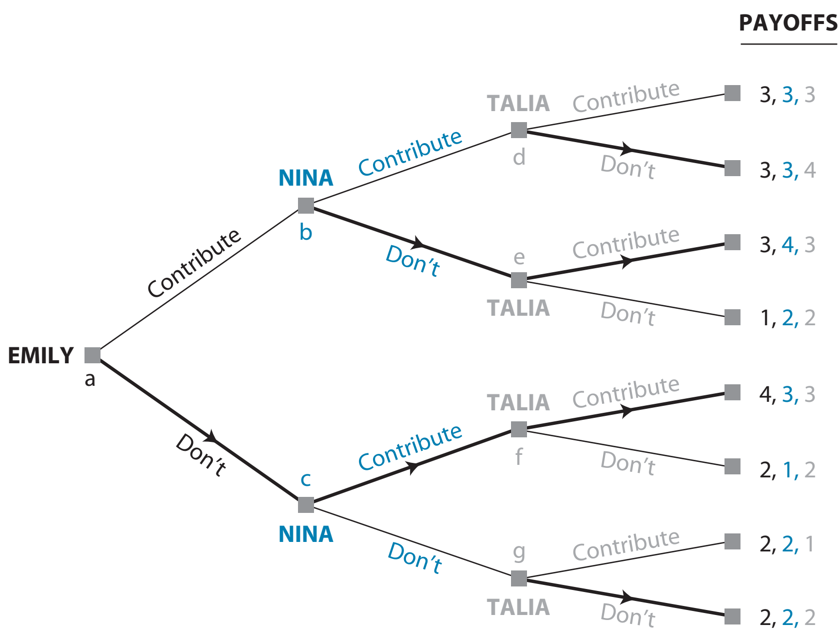


FIGURE 6.10 The Street–Garden Game with Sequential Moves

more simply "DC." Talia has 16 available strategies (choice of two responses at each of four nodes), and her equilibrium strategy is "D following Emily's C and Nina's C, C following their CD, C following their DC, and D following their DD," or "DCCD" for short.

Remember, too, the reason for these choices. The first mover has the opportunity to choose Don't, knowing that the other two will recognize that the nice garden won't be forthcoming unless they contribute and that they like the nice garden sufficiently strongly that they will contribute.

Now we change the rules of the game to make it a simultaneous-move game. (In Chapter 4, we solved a simultaneous-move version with somewhat different payoffs; here we keep the payoffs the same as in Chapter 3.) The payoff matrix is in Figure 6.11. Best-response analysis shows very easily that there are four Nash equilibria.

In three of the Nash equilibria of the simultaneous-move game, two players contribute, while the third does not. These equilibria are similar to the rollback equilibrium of the sequential-move game. In fact, each one corresponds to the rollback equilibrium of the sequential game with a particular order of play. Further, any given order of play in the sequential-move version of this game leads to the same simultaneous-move payoff table.

But there is also a fourth Nash equilibrium here, where no one contributes. *Given* the specified strategies of the other two—namely, Don't contribute—any one player is powerless to bring about the nice garden and therefore chooses not to contribute as well. Thus, in the change from sequential to simultaneous moves, the first-mover advantage has been lost. Multiple equilibria arise, only one of which retains the original first mover's high payoff.

Next we return to the sequential-move version—Emily first, Nina second, and Talia third—but show the game in its normal or strategic form. In the sequential-move game, Emily has 2 pure strategies, Nina has 4, and Talia has 16; so this means constructing a payoff table that is 2 by 4 by 16. With the use of

TALIA chooses:

Contribute

		NINA: Contribute	NINA: Don't
EMILY	Contribute	3, 3, 3	3, 4, 3
	Don't	4, 3, 3	2, 2, 1

Don't Contribute

		NINA: Contribute	NINA: Don't
EMILY	Contribute	3, 3, 4	1, 2, 2
	Don't	2, 1, 2	2, 2, 2

FIGURE 6.11 The Street–Garden Game with Simultaneous Moves

the same conventions as we used for three-player tables in Chapter 4, this particular game would require a table with 16 "pages" of two-by-four payoff tables. That would look too messy; so we opt instead for a reshuffling of the players. Let Talia be the row player, Nina be the column player, and Emily be the page player. Then "all" that is required to illustrate this game is the 16 by 4 by 2 game table shown in Figure 6.12. The order of payoffs still corresponds to our earlier convention in that they are listed row, column, page player; in our example, that means the payoffs are now listed in the order Talia, Nina, and Emily.

As in the monetary–fiscal policy game between the Fed and Congress, there are multiple Nash equilibria in the simultaneous street–garden

EMILY

	Contribute				Don't			
	NINA				**NINA**			
TALIA	CC	CD	DC	DD	CC	CD	DC	DD
CCCC	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CCCD	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
CCDC	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
CDCC	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
DCCC	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CCDD	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2
CDDC	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
DDCC	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CDCD	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
DCDC	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
DCCD	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
CDDD	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2
DCDD	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2
DDCD	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
DDDC	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
DDDD	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2

FIGURE 6.12 Street–Garden Game in Strategic Form

game. (In Exercise S8, we ask you to find them all.) But there is only one subgame-perfect equilibrium, corresponding to the rollback equilibrium found in Figure 6.11. Although best-response analysis does find all of the Nash equilibria, iterated elimination of dominated strategies can reduce the number of reasonable equilibria for us here. This process works because elimination identifies those strategies that include noncredible components (such as "High always" for the Fed in Section 3.B). As it turns out, such elimination can take us all the way to the unique subgame-perfect equilibrium.

In Figure 6.12, we start with Talia and eliminate all of her (weakly) dominated strategies. This step eliminates all but the strategy listed in the eleventh row of the table, DCCD, which we have already identified as Talia's rollback equilibrium strategy. Elimination can continue with Nina, for whom we must compare outcomes from strategies across both pages of the table. To compare her CC to CD, for example, we look at the payoffs associated with CC in *both pages* of the table and compare these payoffs with the similarly identified payoffs for CD. For Nina, the elimination process leaves only her strategy DC; again, this is the rollback equilibrium strategy found for her above. Finally, Emily has only to compare the two remaining cells associated with her choice of Don't and Contribute; she gets the highest payoff when she chooses Don't and so makes that choice. As before, we have identified her rollback equilibrium strategy.

The unique subgame-perfect outcome in the game table in Figure 6.12 thus corresponds to the cell associated with the rollback equilibrium strategies for each player. Note that the process of iterated elimination that leads us to this subgame-perfect equilibrium is carried out by considering the players in reverse order of the actual play of the game. This order conforms to the order in which player actions are considered in rollback analysis and therefore allows us to eliminate exactly those strategies, for each player, that are not consistent with rollback. In so doing, we eliminate all of the Nash equilibria that are not subgame-perfect.

SUMMARY

Many games include multiple components, some of which entail simultaneous play and others of which entail sequential play. In *two-stage* (and multistage) games, a "tree house" can be used to illustrate the game; this construction allows the identification of the different stages of play and the ways in which those stages are linked together. Full-fledged games that arise in later stages of play are called *subgames* of the full game.

Changing the rules of a game to alter the timing of moves may or may not alter the equilibrium outcome of a game. Simultaneous-move games that are changed to make moves sequential may have the same outcome (if both players have dominant strategies), may have a first-mover or second-mover advantage, or may lead to an outcome in which both players are better off. The sequential version of a simultaneous game will generally have a unique rollback equilibrium even if the simultaneous version has no equilibrium or multiple equilibria. Similarly, a sequential-move game that has a unique rollback equilibrium may have several Nash equilibria when the rules are changed to make the game a simultaneous-move game.

Simultaneous-move games can be illustrated in a game tree by collecting decision nodes in *information sets* when players make decisions without knowing at which specific node they find themselves. Similarly, sequential-move games can be illustrated by using a game table; in this case, each player's full set of strategies must be carefully identified. Solving a sequential-move game from its strategic form may lead to many possible Nash equilibria. The number of potential equilibria can be reduced by using the criteria of *credibility* to eliminate some strategies as possible equilibrium strategies. This process leads to the *subgame-perfect equilibrium (SPE)* of the sequential-move game. These solution processes also work for games with additional players.

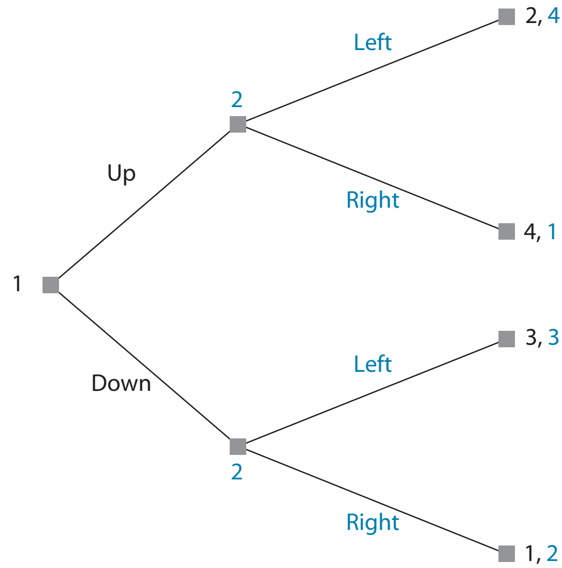
KEY TERMS

continuation (198)
credibility (199)
information set (195)
off-equilibrium path (198)
off-equilibrium subgame (198)
subgame (185)
subgame-perfect equilibrium (SPE) (199)

SOLVED EXERCISES

S1. Consider the simultaneous-move game with two players that has no Nash equilibrium in pure strategies, illustrated in Figure 4.13 in Chapter 4. If the game were transformed into a sequential-move game, would you expect that game to exhibit a first-mover advantage, a second-mover advantage, or neither? Explain your reasoning.

S2. Consider the game represented by the game tree below. The first mover, Player 1, may move either Up or Down, after which Player 2 may move either Left or Right. Payoffs for the possible outcomes appear below. Reexpress this game in strategic (table) form. Find all of the pure-strategy Nash equilibria in the game. If there are multiple equilibria, indicate which one is subgame-perfect. For those equilibria that are not subgame-perfect, identify the reason (the source of the lack of credibility).



S3. Consider the Airbus–Boeing game in Exercise S4 in Chapter 3. Show that game in strategic form and locate all of the Nash equilibria. Which one of the equilibria is subgame-perfect? For those equilibria that are not subgame-perfect, identify the source of the lack of credibility.

S4. Return to the two-player game tree in part (a) of Exercise S2 in Chapter 3.

(a) Write the game in strategic form, making Scarecrow the row player and Tinman the column player.

(b) Find the Nash equilibrium.

S5. Return to the two-player game tree in part (b) of Exercise S2 in Chapter 3.

(a) Write the game in strategic form. (Hint: Refer to your answer to Exercise S2 of Chapter 3.) Find all of the Nash equilibria. There will be many.

(b) For the equilibria that you found in part (a) that are not subgame-perfect, identify the credibility problems.

S6. Return to the three-player game tree in part (c) of Exercise S2 in Chapter 3.

(a) Draw the game table. Make Scarecrow the row player, Tinman the column player, and Lion the page player. (Hint: Refer to your answer to Exercise S2 of Chapter 3.) Find all of the Nash equilibria. There will be many.

(b) For the equilibria that you found in part (a) that are not subgame-perfect, identify the credibility problems.

S7. Consider a simplified baseball game played between a pitcher and a batter. The pitcher chooses between throwing a fastball or a curve, while the batter chooses which pitch to anticipate. The batter has an advantage if he correctly anticipates the type of pitch. In this constant-sum game, the batter's payoff is the probability that the batter will get a base hit. The pitcher's payoff is the probability that the batter fails to get a base hit, which is simply one minus the payoff of the batter. There are four potential outcomes:

(i) If a pitcher throws a fastball, and the batter guesses fastball, the probability of a hit is 0.300.

(ii) If the pitcher throws a fastball, and the batter guesses curve, the probability of a hit is 0.200.

(iii) If the pitcher throws a curve, and the batter guesses curve, the probability of a hit is 0.350.

(iv) If the pitcher throws a curve, and the batter guesses fastball, the probability of a hit is 0.150.

Suppose that the pitcher is "tipping" his pitches. This means that the pitcher is holding the ball, positioning his body, or doing something else in a way that reveals to the batter which pitch he is going to throw. For our purposes, this means that the pitcher-batter game is a sequential game in which the pitcher announces his pitch choice before the batter has to choose his strategy.

(a) Draw this situation, using a game tree.

(b) Suppose that the pitcher knows he is tipping his pitches but can't stop himself from doing so. Thus, the pitcher and batter are playing the game you just drew. Find the rollback equilibrium of this game.

(c) Now change the timing of the game, so that the batter has to reveal his action (perhaps by altering his batting stance) before the pitcher chooses which pitch to throw. Draw the game tree for this situation, and find the rollback equilibrium.

Now assume that the tips of each player occur so quickly that neither opponent can react to them, so that the game is in fact simultaneous.

(d) Draw a game tree to represent this simultaneous game, indicating information sets where appropriate.

(e) Draw the game table for the simultaneous game. Is there a Nash equilibrium in pure strategies? If so, what is it?

S8. The street–garden game analyzed in Section 4 of this chapter has a 16-by-4-by-2 game table when the sequential-move version of the game is expressed in strategic form, as in Figure 6.12. There are *many* Nash equilibria to be found in this table.

(a) Use best-response analysis to find all of the Nash equilibria in the table in Figure 6.12.

(b) Identify the subgame-perfect equilibrium from among your set of all Nash equilibria. Other equilibrium outcomes look identical to the subgame-perfect one—they entail the same payoffs for each of the three players—but they arise after different combinations of strategies. Explain how this can happen. Describe the credibility problems that arise in the nonsubgame-perfect equilibria.

S9. As it appears in the text, Figure 6.1 represents the two-stage game between CrossTalk and GlobalDialog with a combination of tables and trees. Instead, represent the entire two-stage game in a single, very large game tree. Be careful to label which player makes the decision at each node, and remember to draw information sets between nodes where necessary.

S10. Recall the mall location game in Exercise S9 in Chapter 3. That three-player sequential game has a game tree that is similar to the one for the street–garden game, shown in Figure 6.10.

(a) Draw the tree for the mall location game. How many strategies does each store have?

(b) Illustrate the game in strategic form and find all of the pure-strategy Nash equilibria in the game.

(c) Use iterated dominance to find the subgame-perfect equilibrium. (Hint: Reread the last two paragraphs of Section 4.)

S11. The rules of the mall location game, analyzed in Exercise S10 above, specify that when all three stores request space in Urban Mall, the two bigger (more prestigious) stores get the available spaces. The original version of the game also specifies that the firms move sequentially in requesting mall space.

(a) Suppose that the three firms make their location requests simultaneously. Draw the payoff table for this version of the game and find all of the Nash equilibria. Which one of these equilibria do you think is most likely to be played in practice? Explain.

Now suppose that when all three stores simultaneously request Urban Mall, the two spaces are allocated by lottery, giving each store an equal chance of getting into Urban Mall. With such a system, each would have a two-thirds probability (or a 66.67% chance) of getting into Urban Mall when all three had requested space there, and a one-third probability (33.33% chance) of being alone in the Rural Mall.

(b) Draw the game table for this new version of the simultaneous-play mall location game. Find all of the Nash equilibria of the game. Which one of these equilibria do you think is most likely to be played in practice? Explain.

(c) Compare and contrast the equilibria found in part (b) with the equilibria found in part (a). Do you get the same Nash equilibria? Why or why not?

S12. Return to the game of Monica and Nancy in Exercise S10 of Chapter 5. Assume that Monica and Nancy choose their effort levels sequentially instead of simultaneously. Monica commits to her choice of effort first, and on observing this decision, Nancy commits to her own effort.

(a) What is the subgame-perfect equilibrium to the game where the joint profits are $4m + 4n + mn$, the effort costs to Monica and Nancy are m^2 and n^2, respectively, and Monica commits to an effort level first?

(b) Compare the payoffs of Monica and Nancy with those found in Exercise S10 of Chapter 5. Does this game have a first-mover or a second-mover advantage? Explain.

S13. Extending Exercise S12, Monica and Nancy need to decide which (if either) of them will commit to an effort level first. To do this, each of them simultaneously writes on a separate slip of paper whether or not she will commit first. If they both write "yes" or they both write "no," they choose effort levels simultaneously, as in Exercise S10 in Chapter 5. If Monica writes "yes" and Nancy writes "no," then Monica commits to her move first, as in Exercise S12. If Monica writes "no" and Nancy writes "yes," then Nancy commits to her move first.

(a) Use the payoffs to Monica and Nancy in Exercise S12 above as well as in Exercise S10 in Chapter 5 to construct the game table for the first-stage paper-slip decision game. (Hint: Note the symmetry of the game.)

(b) Find the pure-strategy Nash equilibria of this first-stage game.

UNSOLVED EXERCISES

U1. Consider a game in which there are two players, A and B. Player A moves first and chooses either Up or Down. If A chooses Up, the game is over, and each player gets a payoff of 2. If A moves Down, then B gets a turn and chooses between Left and Right. If B chooses Left, both players get 0; if B chooses Right, A gets 3 and B gets 1.

(a) Draw the tree for this game, and find the subgame-perfect equilibrium.

(b) Show this sequential-play game in strategic form, and find all of the Nash equilibria. Which is or are subgame-perfect? Which is or are not? If any are not, explain why.

(c) What method of solution could be used to find the subgame-perfect equilibrium from the strategic form of the game? (Hint: Refer to the last two paragraphs of Section 4.)

U2. Return to the two-player game tree in part (a) of Exercise U2 in Chapter 3.

- **(a)** Write the game in strategic form, making Albus the row player and Minerva the column player. Find all of the Nash equilibria.
- **(b)** For the equilibria you found in part (a) of this exercise that are not subgame-perfect, identify the credibility problems.

U3. Return to the two-player game tree in part (b) of Exercise U2 in Chapter 3.

- **(a)** Write the game in strategic form. Find all of the Nash equilibria.
- **(b)** For the equilibria you found in part (a) that are not subgame-perfect, identify the credibility problems.

U4. Return to the two-player game tree in part (c) of Exercise U2 in Chapter 3.

- **(a)** Draw the game table. Make Albus the row player, Minerva the column player, and Severus the page player. Find all of the Nash equilibria.
- **(b)** For the equilibria you found in part (a) that are not subgame-perfect, identify the credibility problems.

U5. Consider the cola industry, in which Coke and Pepsi are the two dominant firms. (To keep the analysis simple, just forget about all the others.) The market size is $8 billion. Each firm can choose whether to advertise. Advertising costs $1 billion for each firm that chooses it. If one firm advertises and the other doesn't, then the former captures the whole market. If both firms advertise, they split the market 50:50 and pay for the advertising. If neither advertises, they split the market 50:50 but without the expense of advertising.

- **(a)** Write the payoff table for this game, and find the equilibrium when the two firms move simultaneously.
- **(b)** Write the game tree for this game (assume that it is played sequentially), with Coke moving first and Pepsi following.
- **(c)** Is either equilibrium in parts (a) and (b) better from the joint perspective of Coke and Pepsi? How could the two firms do better?

U6. Along a stretch of a beach are 500 children in five clusters of 100 each. (Label the clusters A, B, C, D, and E in that order.) Two ice-cream vendors are deciding simultaneously where to locate. They must choose the exact location of one of the clusters.

If there is a vendor in a cluster, all 100 children in that cluster will buy an ice cream. For clusters without a vendor, 50 of the 100 children are willing to walk to a vendor who is one cluster away, only 20 are willing to walk to a vendor two clusters away, and no children are willing to walk the distance of three or more clusters. The ice cream melts quickly, so the walkers cannot buy for the nonwalkers.

If the two vendors choose the same cluster, each will get a 50% share of the total demand for ice cream. If they choose different clusters, then those children (locals or walkers) for whom one vendor is closer than the other will go to the closer one, and those for whom the two are equidistant will split 50% each. Each vendor seeks to maximize her sales.

(a) Construct the five-by-five payoff table for the vendor location game; the entries stated here will give you a start and a check on your calculations:

> If both vendors choose to locate at A, each sells 85 units.
>
> If the first vendor chooses B and the second chooses C, the first sells 150 and the second sells 170.
>
> If the first vendor chooses E and the second chooses B, the first sells 150 and the second sells 200.

(b) Eliminate dominated strategies as far as possible.

(c) In the remaining table, locate all pure-strategy Nash equilibria.

(d) If the game is altered to one with sequential moves, where the first vendor chooses her location first and the second vendor follows, what are the locations and the sales that result from the subgame-perfect equilibrium? How does the change in the timing of moves here help players resolve the coordination problem in part (c)?

U7. Return to the game among the three lions in the Roman Colosseum in Exercise S8 in Chapter 3.

(a) Write out this game in strategic form. Make Lion 1 the row player, Lion 2 the column player, and Lion 3 the page player.

(b) Find the Nash equilibria for the game. How many did you find?

(c) You should have found Nash equilibria that are not subgame-perfect. For each of those equilibria, which lion is making a noncredible threat? Explain.

U8. Now assume that the mall location game (from Exercises S9 in Chapter 3 and S10 in this chapter) is played sequentially but with a different order of play: Big Giant, then Titan, then Frieda's.

(a) Draw the new game tree.

(b) What is the subgame-perfect equilibrium of the game? How does it compare to the subgame-perfect equilibrium for Exercise S9 in Chapter 3?

(c) Now write the strategic form for this new version of the game.

(d) Find all of the Nash equilibria of the game. How many are there? How does this compare with the number of equilibria from Exercise S10 in this chapter?

U9. Return to the game of Monica and Nancy in Exercise U10 of Chapter 5. Assume that Monica and Nancy choose their effort levels sequentially instead of simultaneously. Monica commits to her choice of effort first. On observing this decision, Nancy commits to her own effort.

(a) What is the subgame-perfect equilibrium to the game where the joint profits are $5m + 4n + mn$, the effort costs to Monica and Nancy are m^2 and n^2, respectively, and Monica commits to an effort level first?

(b) Compare the payoffs of Monica and Nancy with those found in Exercise U10 of Chapter 5. Does this game have a first-mover or second-mover advantage?

(c) Using the same joint profit function as in part (a), find the subgame-perfect equilibrium for the game where *Nancy* must commit first to an effort level.

U10. In an extension of Exercise U9, Monica and Nancy need to decide which (if either) of them will commit to an effort level first. To do this, each of them simultaneously writes on a separate slip of paper whether or not she will commit first. If they both write "yes" or they both write "no," they choose effort levels simultaneously, as in Exercise U10 in Chapter 5. If Monica writes "yes" and Nancy writes "no," they play the game in part (a) of Exercise U9 above. If Monica writes "no" and Nancy writes "yes," they play the game in part (c).

(a) Use the payoffs to Monica and Nancy in parts (b) and (c) in Exercise U9 above, as well as those in Exercise U10 in Chapter 5, to construct the game table for the first-stage paper-slip decision game.

(b) Find the pure-strategy Nash equilibria of this first-stage game.

U11. In the faraway town of Saint James two firms, Bilge and Chem, compete in the soft-drink market (Coke and Pepsi aren't in this market yet). They sell identical products, and since their good is a liquid, they can easily choose to produce fractions of units. Since they are the only two firms in this market, the price of the good (in dollars), P, is determined by $P = (30 - Q_B - Q_C)$, where Q_B is the quantity produced by Bilge and Q_C is the quantity produced by Chem (each measured in liters). At this time both firms are considering whether to invest in new bottling equipment that will lower their variable costs.

(i) If firm j decides *not* to invest, its cost will be $C_j = Q_j^2/2$, where j stands for either B (Bilge) or C (Chem).

(ii) If a firm decides to invest, its cost will be $C_j = 20 + Q_j^2/6$, where j stands for either B (Bilge) or C (Chem). This new cost function reflects the fixed cost of the new machines (20) as well as the lower variable costs.

The two firms make their investment choices simultaneously, but the payoffs in this investment game will depend on the subsequent duopoly games that arise. The game is thus really a two-stage game: decide to invest, and then play a duopoly game.

(a) Suppose both firms decide to invest. Write the profit functions in terms of Q_B and Q_C for the two firms. Use these to find the Nash equilibrium of the quantity-setting game. What are the equilibrium quantities and profits for both firms? What is the market price?

(b) Now suppose both firms decide not to invest. What are the equilibrium quantities and profits for both firms? What is the market price?

(c) Now suppose that Bilge decides to invest, and Chem decides not to invest. What are the equilibrium quantities and profits for both firms? What is the market price?

(d) Write out the two-by-two game table of the investment game between the two firms. Each firm has two strategies: Investment and No Investment. The payoffs are simply the profits found in parts (a), (b), and (c). (Hint: Note the symmetry of the game.)

(e) What is the subgame-perfect equilibrium of the overall two-stage game?

U12. Two French aristocrats, Chevalier Chagrin and Marquis de Renard, fight a duel. Each has a pistol loaded with one bullet. They start 10 steps apart and walk toward each other at the same pace, 1 step at a time. After each step, either may fire his gun. When one shoots, the probability of scoring a hit depends on the distance. After k steps it is $k/5$, and so it rises from 0.2 after the first step to 1 (certainty) after 5 steps, at which point they are right up against one another. If one player fires and misses while the other has yet to fire, the walk must continue even though the bulletless one now faces certain death; this rule is dictated by the code of the aristocracy. Each gets a payoff of -1 if he himself is killed and 1 if the other is killed. If neither or both are killed, each gets 0.

This is a game with five sequential steps and simultaneous moves (shoot or not shoot) at each step. Find the rollback (subgame-perfect) equilibrium of this game.

Hint: Begin at step 5, when the duelists are right up against one another. Set up the two-by-two table for the simultaneous-move game at this step, and find its Nash equilibrium. Now move back to step 4, where the probability of scoring a hit is $4/5$, or 0.8, for each. Set up the two-by-two table for the simultaneous-move game at this step, correctly specifying in the appropriate cell what happens in the future. For example, if one shoots and misses, but the other does not shoot, then the other will wait until step 5 and score a sure hit. If neither shoots, then the game will go to the next step, for which you have already found the equilibrium. Using all this information, find the payoffs in the two-by-two table of step 4, and find the Nash equilibrium at this step. Work backward in the same way through the rest of the steps to find the Nash equilibrium strategies of the full game.

U13. Describe an example of business competition that is similar in structure to the duel in Exercise U12.