15

Strategy and Voting

WHEN MANY OF YOU think about voting, you probably imagine first a national presidential election, then perhaps a local mayoral election, and maybe even an election for class president at your school. But some of you may also be reminded of last year's Heisman Trophy–winning college football player, the latest Academy Award–winning film, or the most recent Supreme Court decision. *All* of these situations involve voting, although they differ based on the number of voters involved, the ballot length or number of choices available to voters, and the procedures used to tally the votes and determine the final winner. In each case, strategic thinking may play a role in how ballots are marked. And strategic considerations can be critical in choosing the method by which votes are taken and then counted.

Voting procedures vary widely not because some votes elect Oscar winners and others elect presidents, but because certain procedures have attributes that make them better (or worse) for specific voting situations. In the past decade, for example, concerns about how elections based on plurality rule (the candidate with the most votes wins) encourage the existence of a two-party political system have led to changes in voting rules in more than a dozen U.S. cities.[1] These changes have led in some cases to election outcomes that differed from those that would have arisen under the old plurality rule system. Jean Quan, the mayor of Oakland, California, for example, won her post in November 2010

[1] This result is known in political science as "Duverger's law." We discuss it in greater detail in Section 3.A.

despite being the first-place choice of only 24% of the voters, while her eventual runner-up had 35% of the first-place votes. In the last round of that city's ranked-choice vote, Quan won 51% of the votes with 49% going to the runner-up. We investigate such paradoxical outcomes in Section 2 of this chapter.

Given the fact that different voting procedures can produce different outcomes, you should immediately see the scope for strategic behavior in choosing a procedure that can generate an outcome you prefer. Perhaps then you can also imagine a situation in which voters might find it beneficial to vote for someone, or something, that is not their top choice in order to avoid having their absolute last choice option be the winner? This type of strategic behavior is common when the voting procedure allows it. As a voter, you should be aware of the benefits associated with such *strategic misrepresentation of preferences* and of the possibility that others may use such tactics against you.

Below, we first introduce you to the range of voting procedures available and to some of the paradoxical outcomes that can arise when specific procedures are used. We then consider how one might judge the performance of those procedures before addressing the strategic behavior of voters and the scope for outcome manipulation. Finally, we present two different versions of a well-known result known as the *median voter theorem*—as a two-person zero-sum game with discrete strategies and with continuous ones.

1 VOTING RULES AND PROCEDURES

Numerous voting procedures are available to help choose from a slate of alternatives (that is, candidates or issues). With as few as three available alternatives, election design becomes interestingly complex. We describe in this section a variety of procedures from three broad classes of voting, or vote-aggregation, methods. The number of possible voting procedures is enormous, and the simple taxonomy that we provide here can be broadened extensively by allowing elections based on a combination of procedures; a considerable literature in both economics and political science deals with just this topic. We have not attempted to provide an exhaustive survey but rather to give a flavor of that literature. If you are interested, we suggest you consult the broader literature for more details on the subject.[2]

[2] The classic textbook on this subject, which was instrumental in making game theory popular in political science, is William Riker, *Liberalism Against Populism* (San Francisco: W. H. Freeman, 1982). A general survey is the symposium on "Economics of Voting," *Journal of Economic Perspectives*, vol. 9, no. 1 (Winter 1995). An important early research contribution is Michael Dummett, *Voting Procedures* (Oxford: Clarendon Press, 1984). Donald Saari, *Chaotic Elections* (Providence, R.I.: American Mathematical Society, 2000), develops some new ideas that we use later in this chapter.

A. Binary Methods

Vote aggregation methods can be classified according to the number of options or candidates considered by the voters at any given time. **Binary methods** require voters to choose between only two alternatives at a time. In elections in which there are exactly two candidates, votes can be aggregated by using the well-known principle of **majority rule**, which simply requires that the alternative with a majority of votes wins. When dealing with a slate of more than two alternatives, **pairwise voting**—a method consisting of a repetition of binary votes—can be used. Pairwise procedures are **multistage**; they entail voting on pairs of alternatives in a series of majority votes to determine which is most preferred.

One pairwise procedure, in which each alternative is put up against each of the others in a round-robin of majority votes, is called the **Condorcet method**, after the eighteenth-century French theorist Jean Antoine Nicholas Caritat, marquis de Condorcet. He suggested that the candidate who defeats each of the others in such a series of one-on-one contests should win the entire election; such a candidate, or alternative, is now termed a **Condorcet winner**. Other pairwise procedures produce "scores" such as the **Copeland index**, which measures an alternative's win-loss record in a round-robin of contests. The first round of the World Cup soccer tournament uses a type of Copeland index to determine which teams from each group move on to the second round of play.[3]

Another well-known pairwise procedure, used when there are three possible alternatives, is the **amendment procedure**, required by the parliamentary rules of the U.S. Congress when legislation is brought to a vote. When a bill is brought before Congress, any amended version of the bill must first win a vote against the original version of the bill. The winner of that vote is then paired against the status quo and members vote on whether to adopt the version of the bill that won the first round; majority rule can then be used to determine the winner. The amendment procedure can be used to consider any three alternatives by pairing two in a first-round election and then putting the third up against the winner in a second-round vote.

B. Plurative Methods

Plurative methods allow voters to consider three or more alternatives simultaneously. One group of plurative voting methods applies information on the positions of alternatives on a voter's ballot to assign points used when tallying ballots; these voting methods are known as **positional methods**. The familiar

[3] Note that such indices, or scores, must have precise mechanisms in place to deal with ties; World Cup soccer uses a system that undervalues a tie to encourage more aggressive play. See Barry Nalebuff and Jonathan Levin, "An Introduction to Vote Counting Schemes," *Journal of Economic Perspectives*, vol. 9, no. 1 (Winter 1995), pp. 3–26.

plurality rule is a special-case positional method in which each voter casts a single vote for her most-preferred alternative. That alternative is assigned a single point when votes are tallied; the alternative with the most votes (or points) wins. Note that a plurality winner need *not* gain a majority, or 51%, of the vote. Thus, for instance, in the 2012 presidential election in Mexico, Enrique Peña Nieto captured the presidency with only 38.2% of the vote; his opponents gained 31.6%, 25.4%, and 2.3% of the vote. Such narrow margins of victory have led to concerns about the legitimacy of past Mexican presidential elections, especially in 2006 when the margin of victory was a mere 0.58 percentage points. Another special-case positional method, the **antiplurality method**, asks voters to vote against one of the available alternatives or, equivalently, to vote for all but one. For counting purposes, the alternative voted against is allocated −1 point, or else all alternatives except that one receive 1 point while the alternative voted against receives 0.

One of the best-known positional methods is the **Borda count**, named after Jean-Charles de Borda, a fellow countryman and contemporary of Condorcet. Borda described the new procedure as an improvement on plurality rule. The Borda count requires voters to rank-order all of the possible alternatives in an election and to indicate their rankings on their ballot cards. Points are assigned to each alternative on the basis of its position on each voter's ballot. In a three-person election, the candidate at the top of a ballot gets 3 points, the next candidate 2 points, and the bottom candidate 1 point. After the ballots are collected, each candidate's points are summed, and the one with the most points wins the election. A Borda count procedure is used in a number of sports-related elections, including professional baseball's Cy Young Award and college football's championship elections.

Many other positional methods can be devised simply by altering the rule used for the allocation of points to alternatives based on their positions on a voter's ballot. One system might allocate points in such a way as to give the top-ranked alternative relatively more than the others—for example, 5 points for the most-preferred alternative in a three-way election but only 2 and 1 for the second- and third-ranked options. In elections with larger numbers of candidates—say, eight—the top two choices on a voter's ballot might receive preferred treatment, gaining 10 and 9 points, respectively, while the others receive 6 or fewer.

An alternative to the positional plurative methods is the relatively recently invented **approval voting** method, which allows voters to cast a single vote for each alternative of which they "approve."[4] Unlike positional methods, approval voting does not distinguish between alternatives on the basis of their positions

[4] Unlike many of the other methods that have histories going back several centuries, the approval voting method was designed and named by then–graduate student Robert Weber in 1971; Weber is now a professor of managerial economics and decision sciences at Northwestern University, specializing in game theory.

on the ballot. Rather, all approval votes are treated equally, and the alternative that receives the most approvals wins. In elections in which more than one winner can be selected (in electing a school board, for instance), a threshold level of approvals is set in advance, and alternatives with more than the required minimum approvals are elected. Proponents of this method argue that it favors relatively moderate alternatives over those at either end of the spectrum; opponents claim that unwary voters could elect an unwanted novice candidate by indicating too many "encouragement" approvals on their ballots. Despite these disagreements, several professional societies and the United Nations have adopted approval voting to elect their officers, and some states have used or are considering using this method for public elections.

C. Mixed Methods

Some multistage voting procedures combine plurative and binary voting in **mixed methods**. The **majority runoff** procedure, for instance, is a two-stage method used to decrease a large group of possibilities to a binary decision. In a first-stage election, voters indicate their most-preferred alternative, and these votes are tallied. If one candidate receives a majority of votes in the first stage, she wins. However, if there is no majority choice, a second-stage election pits the two most-preferred alternatives against each other. Majority rule chooses the winner in the second stage. French presidential elections use the majority runoff procedure, which can yield unexpected results if three or four strong candidates split the vote in the first round. In the spring of 2002, for example, the far-right candidate Le Pen came in second ahead of France's socialist Prime Minister Jospin in the first round of the presidential election. This result aroused surprise and consternation among French citizens, 30% of whom hadn't even bothered to vote in the election and some of whom had taken the first round as an opportunity to express their preference for various candidates of the far and fringe left. Le Pen's advance to the runoff election led to considerable political upheaval, although he lost in the end to the incumbent president, Chirac.

Another mixed procedure consists of voting in successive **rounds**. Voters consider a number of alternatives in each round of voting, with the worst-performing alternative eliminated after each stage. Voters then consider the remaining alternatives in a next round. The elimination continues until only two alternatives remain; at that stage, the method becomes binary, and a final majority runoff determines a winner. A procedure with rounds is used to choose sites for the Olympic Games.

One could eliminate the need for successive rounds of voting by having voters indicate their preference orderings on the first ballot. Then a **single transferable vote** method can be used to tally votes in later rounds. With a single transferable vote, each voter indicates her preference by ordering all candidates on a single initial ballot. If no alternative receives a majority of all first-place votes, the bottom-ranked alternative is eliminated and all first-place votes for that candidate are "transferred" to the candidate listed second on those ballots; similar reallocation occurs in later rounds as additional alternatives are eliminated until a majority winner emerges. This voting method, more commonly called **instant runoff**, is now used in over a dozen U.S. cities, including Oakland and San Francisco. Some cities have begun calling it **rank-choice voting** due to voter expectations of "instant" results from a procedure that actually requires as many as two to three days for full ballot-counting to be completed.

The single transferable vote is sometimes combined with **proportional representation** in an election. Proportional representation implies that a state electorate consisting of 55% Republicans, 25% Democrats, and 20% Independents, for example, would yield a body of representatives mirroring the party affiliations of that electorate. In other words, 55% of the U.S. Representatives from such a state would be Republican, and so on; this result contrasts starkly with the plurality rule method, which would elect *all* Republicans (assuming that the voter mix in each district exactly mirrors the overall voter mix in the state). Candidates who attain a certain quota of votes are elected, and others who fall below a certain quota are eliminated, depending on the exact specifications of the voting procedure. Votes for those candidates who are eliminated are again transferred by using the voters' preference orderings. This procedure continues until an appropriate number of candidates from each party is elected. Versions of this type of procedure are used in parliamentary elections in both Australia and New Zealand.

Clearly, there is room for considerable strategic thinking in the choice of a vote aggregation method, and strategy is also important even after the rule has been chosen. We examine some of the issues related to rule making and agenda setting in Section 2. Furthermore, strategic behavior on the part of voters, often called **strategic voting** or **strategic misrepresentation of preferences**, can also alter election outcomes under any set of rules, as we will see later in this chapter.

2 VOTING PARADOXES

Even when people vote according to their true preferences, specific conditions on voter preferences and voting procedures can give rise to curious outcomes. In addition, election outcomes can depend critically on the type of procedure used to aggregate votes. This section describes some of the most famous of the curious outcomes—the so-called voting paradoxes—as well as some examples of how election results can change under different vote-aggregation methods with no change in voter preferences and no strategic voting.

A. The Condorcet Paradox

The **Condorcet paradox** is one of the most famous and important of the voting paradoxes.[5] As mentioned earlier, the Condorcet method calls for the winner to be the candidate who gains a majority of votes in each round of a round-robin of pairwise comparisons. The paradox arises when no Condorcet winner emerges from this process.

To illustrate the paradox, we construct an example in which three people vote on three alternative outcomes by using the Condorcet method. Consider three city councillors (Left, Center, and Right) who are asked to rank their preferences for three alternative welfare policies, one that extends the welfare benefits currently available (call this one Generous, or G), another that decreases available benefits (Decreased, or D), and yet another that maintains the status quo (Average, or A). They are then asked to vote on each pair of policies to establish a council ranking, or a **social ranking**. This ranking is meant to describe how the council as a whole judges the merits of the possible welfare systems.

Suppose Councillor Left prefers to keep benefits as high as possible, whereas Councillor Center is most willing to maintain the status quo but concerned about the state of the city budget and so least willing to extend welfare benefits. Finally, Councillor Right most prefers reducing benefits but prefers an increase in benefits to the status quo; she expects that extending benefits will soon cause a serious budget crisis and turn public opinion so much against benefits that a more permanent state of low benefits will result, whereas the status quo could go on indefinitely. We illustrate these preference orderings in Figure 15.1 where the "curly" greater-than symbol, $\succ$, is used to indicate that one alternative is preferred to another. (Technically, $\succ$ is referred to as a *binary ordering relation*.)

With these preferences, if Generous is paired against Average, Generous wins. In the next pairing, of Average against Decreased, Average wins. And in the final pairing of Generous against Decreased, the vote is again 2 to 1, this time in favor of Decreased. Therefore, if the council votes on alternative pairs of policies, a majority prefer Generous over Average, Average over Decreased, *and* Decreased over Generous. No one policy has a majority over both of the others. The group's preferences are cyclical: $G \succ A \succ D \succ G$.

LEFT	CENTER	RIGHT
$G \succ A \succ D$	$A \succ D \succ G$	$D \succ G \succ A$

FIGURE 15.1 Councillor Preferences over Welfare Policies

[5] It is so famous that economists have been known to refer to it as *the* voting paradox. Political scientists appear to know better, in that they are far more likely to use its formal name. As we will see, there are any number of possible voting paradoxes, not just the one named for Condorcet.

This cycle of preferences is an example of an **intransitive ordering** of preferences. The concept of rationality is usually taken to mean that individual preference orderings are **transitive** (the opposite of intransitive). If someone is given choices A, B, and C and you know that she prefers A to B and B to C, then transitivity implies that she also prefers A to C. (The terminology comes from the transitivity of numbers in mathematics; for instance, if $3 > 2$ and $2 > 1$, then we know that $3 > 1$.) A transitive preference ordering will not cycle as does the social ordering derived in our city council example; hence, we say that such an ordering is intransitive.

Notice that all of the *councillors* have transitive preferences over the three welfare policy alternatives but the *council* does not. This is the Condorcet paradox: even if all individual preference orderings are transitive, there is no guarantee that the social-preference ordering induced by Condorcet's voting procedure also will be transitive. The result has far-reaching implications for public servants, as well as for the general public. It calls into question the basic notion of the "public interest," because such interests may not be easily defined or may not even exist. Our city council does not have any well-defined set of group preferences over the welfare policies. The lesson is that societies, institutions, or other large groups of people should not always be analyzed as if they acted like individuals.

The Condorcet paradox can even arise more generally. There is no guarantee that the social ordering induced by *any* formal group-voting process will be transitive just because individual preferences are. However, some estimates have shown that the paradox is most likely to arise when large groups of people are considering large numbers of alternatives. Smaller groups considering smaller numbers of alternatives are more likely to have similar preferences over those alternatives; in such situations, the paradox is much less likely to appear.[6] In fact, the paradox arose in our example because the council completely disagreed not only about which alternative was best but also about which was worst. The smaller the group, the less likely such outcomes are to occur.

B. The Agenda Paradox

The second paradox that we consider also entails a binary voting procedure, but this example considers the ordering of alternatives in that procedure. In a parliamentary setting with a committee chair who determines the specific order of voting for a three-alternative election, substantial power over the final outcome lies with the chair. In fact, the chair can take advantage of the intransitive social-preference ordering that arises from some sets of individual preferences and, by selecting an appropriate agenda, manipulate the outcome of the election in any manner she desires.

[6] See Peter Ordeshook, *Game Theory and Political Theory* (Cambridge: Cambridge University Press, 1986), p. 58.

Consider again the city councillors Left, Center, and Right, who must decide among Generous, Average, and Decreased welfare policies. The councillors' preferences over the alternatives were shown in Figure 15.1. Let us now suppose that one of the councillors has been appointed chair of the council by the mayor, and the chair is given the right to decide which two welfare policies get voted on first and which goes up against the winner of that initial vote. With the given set of councillor preferences and common knowledge of the preference orderings, the chair can get any outcome that she wants. If Left were chosen chair, for example, she could orchestrate a win for Generous by setting Average against Decreased in the first round, with the winner to go up against Generous in round two. The result that any final ordering can be obtained by choosing an appropriate procedure is known as the **agenda paradox**.

The only determinant of the outcome in our city council example is the ordering of the agenda. Setting the agenda is the real game here, and because the chair sets the agenda, the appointment or election of the chair is the true outlet for strategic behavior. Here, as in many other strategic situations, what appears to be the game (in this case, choice of a welfare policy) is not the true game at all; rather, those participating in the game engage in strategic play at an earlier point (deciding the identity of the chair) and vote according to set preferences in the eventual election.

However, the preceding demonstration of the agenda setter's power assumes that in the first round, voters choose between the two alternatives (Average and Decreased) on the basis only of their preferences between these two alternatives, with no regard for the eventual outcome of the procedure. Such behavior is called **sincere voting**; actually, myopic or nonstrategic voting would be a better name. If Center is a strategic game player, she should realize that if she votes for Decreased in the first round (even though she prefers Average between the pair presented at that stage), then Decreased will win the first round and will also win against Generous in the second round with support from Right. Center prefers Decreased over Generous as the eventual outcome. Therefore, she should do this rollback analysis and vote strategically in the first round. But should she, if everyone else is also voting strategically? We examine the game of strategic voting and find its equilibrium in Section 4.

C. The Reversal Paradox

Positional voting methods also can lead to paradoxical results. The Borda count, for example, can yield the **reversal paradox** when the slate of candidates open to voters changes. This paradox arises in an election with at least four alternatives when one of them is removed from consideration after votes have been submitted, making recalculation necessary.

Suppose there are four candidates for a (hypothetical) special commemorative Cy Young Award to be given to a retired major-league baseball pitcher.

ORDERING 1 (2 voters)	ORDERING 2 (3 voters)	ORDERING 3 (2 voters)
Koufax > Seaver > Roberts > Carlton	Carlton > Koufax > Seaver > Roberts	Seaver > Roberts > Carlton > Koufax

FIGURE 15.2 Sportswriter Preferences over Pitchers

The candidates are Steve Carlton (SC), Sandy Koufax (SK), Robin Roberts (RR), and Tom Seaver (TS). Seven prominent sportswriters are asked to rank these pitchers on their ballot cards. The top-ranked candidate on each card will get 4 points; decreasing numbers of points will be allotted to candidates ranked second, third, and fourth.

Across the seven voting sportswriters, there are three different preference orderings over the candidate pitchers; these preference orderings, with the number of writers having each ordering, are shown in Figure 15.2. When the votes are tallied, Seaver gets $(2 \times 3) + (3 \times 2) + (2 \times 4) = 20$ points; Koufax gets $(2 \times 4) + (3 \times 3) + (2 \times 1) = 19$ points; Carlton gets $(2 \times 1) + (3 \times 4) + (2 \times 2) = 18$ points; and Roberts gets $(2 \times 2) + (3 \times 1) + (2 \times 3) = 13$ points. Seaver wins the election, followed by Koufax, Carlton, and Roberts in last place.

Now suppose it is discovered that Roberts is not really eligible for the commemorative award, because he never actually won a Cy Young Award, having reached the pinnacle of his career in the years just before the institution of the award in 1956. This discovery requires points to be recalculated, ignoring Roberts on the ballots. The top spot on each card now gets 3 points, while the second and third spots receive 2 and 1, respectively. Ballots from sportswriters with preference ordering 1, for example, now give Koufax and Seaver 3 and 2 points, respectively, rather than 4 and 3 from the first calculation; those ballots also give Carlton a single point for last place.

Adding votes with the revised point system shows that Carlton receives 15 points, Koufax receives 14 points, and Seaver receives 13 points. Winner has turned loser as the new results reverse the standings found in the first election. No change in preference orderings accompanies this result. The only difference in the two elections is the number of candidates being considered. In Section 3, we identify the key vote-aggregation principle violated by the Borda count that leads to the reversal paradox.

D. Change the Voting Method, Change the Outcome

As should be clear from the preceding discussion, election outcomes are likely to differ under different sets of voting rules. As an example, consider 100 voters who can be broken down into three groups on the basis of their preferences over

GROUP 1 (40 voters)	GROUP 2 (25 voters)	GROUP 3 (35 voters)
A > B > C	B > C > A	C > B > A

FIGURE 15.3 Group Preferences over Candidates

three candidates (A, B, and C). Preferences of the three groups are shown in Figure 15.3. With the preferences as shown, and depending on the vote-aggregation method used, any of these three candidates could win the election.

With simple plurality rule, candidate A wins with 40% of the vote, even though 60% of the voters rank her lowest of the three. Supporters of candidate A would obviously prefer this type of election. If they had the power to choose the voting method, then plurality rule, a seemingly "fair" procedure, would win the election for A in spite of the majority's strong dislike for that candidate.

The Borda count, however, would produce a different outcome. In a Borda system with 3 points going to the most-preferred candidate, 2 points to the middle candidate, and 1 to the least-preferred candidate, A gets 40 first-place votes and 60 third-place votes, for a total of $40(3) + 60(1) = 180$ points. Candidate B gets 25 first-place votes and 75 second-place votes, for a total of $25(3) + 75(2) = 225$ points; and C gets 35 first-place votes, 25 second-place votes, and 40 third-place votes, for a total of $35(3) + 25(2) + 40(1) = 195$ points. In this procedure, B wins, with C in second place and A last. Candidate B would also win with the antiplurality vote, in which electors cast votes for all but their least-preferred candidate.

And what about candidate C? She can win the election if a majority or an instant-runoff system is used. In either method, A and C, with 40 and 35 votes in the first round, survive to face each other in the runoff. The majority-runoff system would call voters back to the polls to consider A and C; the instant runoff system would eliminate B and reallocate B's votes (from group 2 voters) to the next preferred alternative, candidate C. Then, because A is the least-preferred alternative for 60 of the 100 voters, candidate C would win the runoff election 60 to 40.

Another example of how different procedures can lead to different outcomes can be seen in the case of the 2010 Oakland mayoral election described in the introduction to this chapter. Olympics site-selection voting is now done using instant runoff instead of several rounds of plurality rule with elimination after similar unusual results in voting for the 1996 and 2000 host cities. In both cases, the plurality winner in all but the penultimate round lost to the one remaining rival city in the last round. Athens lost out to Atlanta for the 1996 Games and Beijing lost out to Sydney for the 2000 Games.

3 EVALUATING VOTING SYSTEMS

The discussion of the various voting paradoxes in Section 2 suggests that voting methods can suffer from a number of faults that lead to unusual, unexpected, or even unfair outcomes. In addition, this suggestion leads us to ask: Is there one voting system that satisfies certain regularity conditions, including transitivity, and that is the most "fair"—that is, most accurately captures the preferences of the electorate? Kenneth Arrow's **impossibility theorem** tells us that the answer to this question is no.[7]

The technical content of Arrow's theorem makes it beyond our scope to prove completely. But the sense of the theorem is straightforward. Arrow argued that no preference-aggregation method could satisfy all six of the critical principles that he identified:

1. The social or group ranking must rank all alternatives (be complete).
2. It must be transitive.
3. It should satisfy a condition known as *positive responsiveness*, or the Pareto property. Given two alternatives, A and B, if the electorate unanimously prefers A to B, then the aggregate ranking should place A above B.
4. The ranking must not be imposed by external considerations (such as customs) independent of the preferences of individual members of the society.
5. It must not be dictatorial—no single voter should determine the group ranking.
6. And it should be independent of irrelevant alternatives; that is, no change in the set of candidates (addition to or subtraction from) should change the rankings of the unaffected candidates.

Often, the theorem is abbreviated by imposing the first four conditions and focusing on the difficulty of simultaneously obtaining the last two; the simplified form states that we cannot have independence of irrelevant alternatives (IIA) without dictatorship.[8]

You should be able to see immediately that some of the voting methods considered earlier do not satisfy all of Arrow's principles. The requirement of IIA, for example, is violated by the single transferable-vote procedure as well as by the Borda count, as we saw in Section 2.C. Borda's procedure is, however, nondictatorial and consistent, and it satisfies the Pareto property. All of the other systems that we have considered satisfy IIA but break down on one of the other principles.

[7] A full description of this theorem, often called "Arrow's General Possibility Theorem," can be found in Kenneth Arrow, *Social Choice and Individual Values*, 2nd ed. (New York: Wiley, 1963).

[8] See Nicholson and Snyder's treatment of Arrow's impossibility theorem in their *Microeconomic Theory*, 11th ed. (New York: Cengage Learning, 2012), ch. 19, for more detail at a level appropriate for intermediate-level economics students.

Arrow's theorem has provoked extensive research into the robustness of his conclusion to changes in the underlying assumptions. Economists, political scientists, and mathematicians have searched for a way to reduce the number of criteria or relax Arrow's principles minimally to find a procedure that satisfies the criteria without sacrificing the core principles; their efforts have been largely unsuccessful. Most economic and political theorists now accept the idea that some form of compromise is necessary when choosing a vote- or preference-aggregation method. Here are a few prominent examples, each representing the approach of a particular field—political science, economics, and mathematics.

A. Black's Condition

As the discussion in Section 2.A showed, the pairwise voting procedure does not satisfy Arrow's condition on transitivity of the social ranking, even when all individual rankings are transitive. One way to surmount this obstacle to meeting Arrow's conditions, as well as a way to prevent the Condorcet paradox, is to place restrictions on the preference orderings held by individual voters. Such a restriction, known as the requirement of **single-peaked preferences**, was put forth by the political scientist Duncan Black in the late 1940s.[9] Black's seminal paper on group decision making actually predates Arrow's impossibility theorem and was formulated with the Condorcet paradox in mind, but voting theorists have since shown its relevance to Arrow's work; in fact, the requirement of single-peaked preferences is sometimes referred to as **Black's condition**.

For a preference ordering to be single peaked, it must be the case that the alternatives being considered can be ordered along some specific dimension (for example, the expenditure level associated with each policy). To illustrate this requirement, we draw a graph in Figure 15.4 with the specified dimension on the horizontal axis and a voter's preference ranking (or payoff) on the vertical axis. For the single-peaked requirement to hold, each voter must have a single ideal or

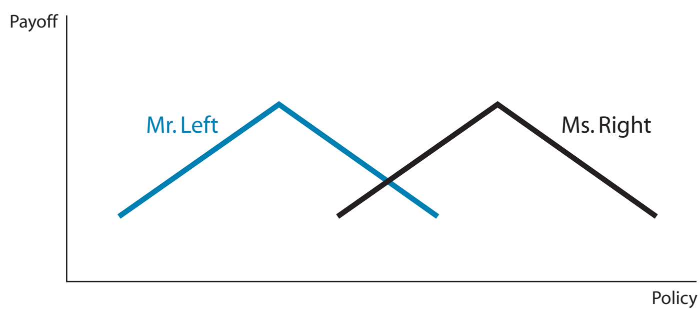

FIGURE 15.4 Single-Peaked Preferences

[9] Duncan Black, "On the Rationale of Group Decision-Making," *Journal of Political Economy*, vol. 56, no. 1 (February 1948), pp. 23–34.

most-preferred alternative, and alternatives "farther away" from the most-preferred point must provide steadily lower payoffs. The two voters in Figure 15.4, Mr. Left and Ms. Right, have different ideal points along the policy dimension, but for each, the payoff falls steadily as the policy moves away from his or her ideal point.

Black shows that if preferences of each voter are single peaked, then the pairwise (majority) voting procedure must produce a transitive social ordering. The Condorcet paradox is prevented, and pairwise voting satisfies Arrow's transitivity condition.

B. Robustness

An alternative, more recent method of compromise with Arrow comes from the economic theorists Partha Dasgupta and Eric Maskin.[10] They suggest a new criterion called **robustness** by which to judge voting methods. Robustness is measured by considering how often a voting procedure that is nondictatorial and that satisfies IIA as well as the Pareto property also satisfies the requirement of transitivity of its social ranking: For how many sets of voter-preference orderings does such a procedure satisfy transitivity?

With the use of the robustness criterion, simple majority rule can be shown to be *maximally robust*—that is, it is nondictatorial, satisfies IIA and Pareto, and provides transitive social rankings for the largest possible set of voter-preference orderings. Behind majority rule on the robustness scale lie other voting procedures, including the Borda count and plurality rule. The robustness criterion is appealing in its ability to establish one of the most commonly used voting procedures—the one most often associated with the democratic process—as a candidate for the best aggregation procedure.

C. Intensity Ranking

Another class of attempts to escape from Arrow's negative result focuses on the difficulty of satisfying Arrow's IIA requirement. A recent theory of this kind comes from the mathematician Donald Saari.[11] He suggests that a vote-aggregation method might use more information about voters' preferences than is contained in their mere ordering of any pair of alternatives, X and Y; rather, it could take into account each individual voter's *intensity* of preferences between that pair of alternatives. This intensity can be measured by counting the number of other

[10] See Partha Dasgupta and Eric Maskin, "On the Robustness of Majority Rule," *Journal of the European Economic Association*, vol. 6 (2008), pp. 949–73.

[11] For more precise information about Saari's work on Arrow's theorem, see D. Saari, "Mathematical Structure of Voting Paradoxes I: Pairwise Vote," *Economic Theory*, vol. 15 (2000), pp. 1–53. Additional information on this result and on the robustness of the Borda count can be found in D. Saari, *Chaotic Elections* (Providence, R.I.: American Mathematical Society, 2000).

alternatives, Z, W, V, . . . that a voter places between X and Y. Saari therefore replaces the IIA condition, number 6 of Arrow's principles, with a different one, which he labels IBI (intensity of binary independence) and which we will number 6′:

6′. Society's relative ranking of any two alternatives should be determined only by (1) each voter's relative ranking of the pair and (2) the intensity of this ranking.

This condition is weaker than IIA because it effectively applies IIA only to such additions or deletions of "irrelevant" alternatives that do not change the intensity of people's preferences between the "relevant" ones. With this revision, the Borda count satisfies the modified Arrow theorem. It is the only one of the positional voting methods that does so.

Saari also hails the Borda count as the only procedure that appropriately observes ties within collections of ballots, a criterion that he argues is essential for a good aggregation system to satisfy. Ties can occur two ways: through **Condorcet terms** or through **reversal terms** within voter-preference orderings. In a three-candidate election among alternatives A, B, and C, the Condorcet terms are the preference orderings A > B > C, B > C > A, and C > A > B. A set of three ballots with these preferences appearing on one ballot apiece should logically offset each other, or constitute a tie. Reversal terms are preference orderings that contain a reversal in the location of a *pair* of alternatives. In the same election, two ballots with preference orderings of A > B > C and B > A > C should logically lead to a tie in a pairwise contest between A and B. Only the Borda procedure treats collections of ballots with Condorcet terms or reversal terms as tied. Although the Borda count can lead to the reversal paradox, as shown in the preceding section, it retains many proponents. The *only* time that the Borda procedure produces paradoxical results is when alternatives are dropped from consideration after ballots have been collected. Because such results can be prevented by using only ballots for the complete set of final candidates, the Borda procedure has gained favor in some circles as one of the best vote-aggregation methods.

Other researchers have made different suggestions regarding criteria that a good aggregation system should satisfy. Some of them include the *Condorcet criterion* (that a Condorcet winner should be selected by a voting system, if such a winner exists), the *consistency criterion* (that an election including all voters should elect the same alternative as would two elections held for an arbitrary division of the entire set of voters), and lack of manipulability (a voting system should not encourage manipulability—strategic voting—on the part of voters). We cannot consider each of these suggestions at length, but we do address strategic manipulation by voters in the following section.

4 STRATEGIC MANIPULATION OF VOTES

Several of the voting systems that we have considered yield considerable scope for strategic misrepresentation of preferences by voters. In Section 2.B, we showed how the power of an agenda-setting Left chair can be countered by a Center councillor voting in the first round against her true preference, so as to knock out her least-preferred alternative and send a more preferred one into the second round. More generally, voters can choose to vote for candidates, issues, or policies that are not actually their most-preferred outcomes among the alternatives presented in an early round if such behavior can alter the final election results in their favor. In this section, we consider a number of ways in which strategic voting behavior can affect elections.

A. Plurality Rule

Plurality-rule elections, often perceived as the fairest by many voters, still provide opportunities for strategic behavior. In presidential elections, for instance, there are generally two major candidates in contention. When such a race is relatively close, there is potential for a third candidate to enter the race and divert votes away from the leading candidate; if the entry of this third player truly threatens the chances of the leader winning the election, the late entrant is called a **spoiler**.

Spoilers are generally believed to have little chance to win the whole election, but their role in changing the election outcome is undisputed. In elections with a spoiler candidate, those who prefer the spoiler to the leading major candidate but least prefer the trailing major candidate may do best to strategically misrepresent their preferences to prevent the election of their least-favorite candidate. That is, you should vote for the leader in such a case even though you would prefer the spoiler because the spoiler is unlikely to garner a plurality; voting for the leader then prevents the trailing candidate, your least favorite, from winning.[12] Ross Perot played such a role in the 1992 U.S. presidential election, apparently succumbing to defeat due to misrepresented preferences. A *Newsweek* survey claimed that if more voters had believed Perot was capable of winning the election, he might have done so; a plurality of 40% of voters surveyed said they would have voted for Perot (instead of Bush or Clinton) if they had thought he could have won.[13]

[12] Note that an approval-voting method would not suffer from this same problem.

[13] "Ross Reruns," *Newsweek*, Special Election Recap Issue, November 18, 1996, p. 104.

Ralph Nader played a similar role in the 2000 presidential election, although he was more concerned about garnering 5% of the popular vote so that his Green Party could qualify for federal matching election funds than he was about actually winning the presidency. Because Nader was pulling needed votes from Democrat Al Gore's supporters, several groups (as well as a number of Web sites) advocated "vote swapping" schemes designed to gain Nader his needed votes without costing Gore the electoral votes of any of his key states. Nader voters in key Gore states (such as Pennsylvania, Michigan, and Maine) were asked to "swap" their votes with Gore supporters in a state destined to go to George W. Bush (such as Texas or Wyoming); a Michigan Nader supporter could vote for Gore while her Nader vote was cast in Texas. Evidence on the efficacy of these strategies is mixed. We do know that Nader failed to win his 5% of the popular vote but that Gore carried all of Pennsylvania, Michigan, and Maine.

In elections for legislatures, where many candidates are chosen, the performance of third parties is very different under a system of proportional representation of the whole population in the whole legislature from that under a system of plurality in separate constituencies. Britain has the constituency and plurality system. In the past 50 years, the Labor and Conservative parties have shared power. The Liberal Party, despite sizable third-place support in the electorate, has suffered from strategic voting and therefore has had disproportionately few seats in Parliament. Italy has had the nationwide list and proportional representation system; there is no need to vote strategically in such a system, and even small parties can have significant presence in the legislature. Often, no party has a clear majority of seats, and small parties can affect policy through bargaining for alliances.

A party cannot flourish if it is largely ineffective in influencing a country's political choices. Therefore, we tend to see just two major parties in countries with the plurality system and several parties in those with the proportional representation system. Political scientists call this observation *Duverger's law*.

In the legislature, the constituency system tends to produce only two major parties—often one of them with a clear majority of seats and therefore more decisive government. But it runs the risk that the minority's interests will be overlooked—that is, of producing a "tyranny of the majority." A proportional representation system gives more of a voice to minority views. But it can produce inconclusive bargaining for power and legislative gridlock. Interestingly, each country seems to believe that its system performs worse and considers switching to the other; in Britain, there are strong voices calling for proportional representation, and Italy has been seriously considering a constituency system.

B. Pairwise Voting

When you know that you are bound by a pairwise method such as the amendment procedure, you can use your prediction of the second-round outcome to determine your optimal voting strategy in the first round. It may be in your interest to appear committed to a particular candidate or policy in the first round, even if it is not your most-preferred alternative, so that your least-favorite alternative cannot win the entire election in the second round.

We return here to our example of the city council with an agenda-setting chair; again, all three preference rankings are assumed to be known to the entire council. Suppose Councillor Left, who most prefers the Generous welfare package, is appointed chair and sets the Average and Decreased policies against each other in a first vote, with the winner facing off against the Generous policy in the second round. If the three councillors vote strictly according to their preferences, shown in Figure 15.1, Average will beat Decreased in the first vote and Generous will then beat Average in the second vote; the chair's preferred outcome will be chosen. The city councillors are likely to be well-trained strategists, however, who can look ahead to the final round of voting and use rollback to determine which way to vote in the opening round.

In the scenario just described, Councillor Center's least-preferred policy will be chosen in the election. Therefore, rollback analysis says that she should vote strategically in the first round to alter the election's outcome. If Center votes for her most-preferred policy in the first round, she will vote for the Average policy, which will then beat Decreased in that round and lose to Generous in round two. However, she could instead vote strategically for the Decreased policy in the first round, which would lift Decreased over Average on the first vote. Then, when Decreased is set up against Generous in the second round, Generous will lose to Decreased. Councillor Center's misrepresentation of her preference ordering with respect to Average and Decreased helps her to change the winner of the election from Generous to Decreased. Although Decreased is not her most-preferred outcome, it is better than Generous from her perspective.

This strategy works well for Center if she can be sure that no other strategic votes will be cast in the election. Thus, we need to analyze both rounds of voting fully to verify the Nash equilibrium strategies for the three councillors. We do so by using rollback on the two simultaneous-vote rounds of the election, starting with the two possible second-round contests, A versus G or D versus G. In the following analysis, we use the abbreviated names of the policies, G, A, and D.

Figure 15.5 illustrates the outcomes that arise in each of the possible second-round elections. The two tables in Figure 15.5a show the winning policy (not payoffs to the players) when A has won the first round and is pitted against G; the tables in Figure 15.5b show the winning policy when L has won the first

(a) A versus G election

Right votes:

A

		CENTER	
		A	G
LEFT	A	A	A
	G	A	G

G

		CENTER	
		A	G
LEFT	A	A	G
	G	G	G

(b) D versus G election

Right votes:

D

		CENTER	
		D	G
LEFT	D	D	D
	G	D	G

G

		CENTER	
		D	G
LEFT	D	D	G
	G	G	G

FIGURE 15.5 Election Outcomes in Two Possible Second-Round Votes

round. In both cases, Councillor Left chooses the row of the final outcome, Center chooses the column, and Right chooses the actual table (left or right).

You should be able to establish that each councillor has a dominant strategy in each second-round election. In the A-versus-G election, Left's dominant strategy is to vote for G, Center's dominant strategy is to vote for A, and Right's dominant strategy is to vote for G; G will win this election. If the councillors consider D versus G, Left's dominant strategy is still to vote for G, and Right and Center both have a dominant strategy to vote for D; in this vote, D wins. A quick check shows that all of the councillors vote according to their true preferences in this round. Thus, these dominant strategies are all the same: "Vote for the alternative that I prefer." Because there is no future to consider in the second-round vote, the councillors simply vote for whichever policy ranks higher in their preference ordering.[14]

We can now use the results from our analysis of Figure 15.5 to consider optimal voting strategies in the first round of the election, in which voters choose

[14] It is a general result in the voting literature that voters faced with pairs of alternatives will always vote truthfully at the last round of voting.

Right votes:

A

		CENTER A	CENTER D
LEFT	A	G	G
LEFT	D	G	D

D

		CENTER A	CENTER D
LEFT	A	G	D
LEFT	D	D	D

FIGURE 15.6 Election Outcomes Based on First-Round Votes

between policies A and D. Because we know how the councillors will vote in the next round given the winner here, we can show the outcome of the entire election in the tables in Figure 15.6.

As an example of how we arrived at these outcomes, consider the G in the upper-left-hand cell of the right-hand table in Figure 15.6. The outcome in that cell is obtained when Left and Center both vote for A in the first round while Right votes for D. Thus, A and G are paired in the second round, and as we saw in Figure 15.5, G wins. The other outcomes are derived in similar fashion.

Given the outcomes in Figure 15.6, Councillor Left (who is the chair and has set the agenda) has a dominant strategy to vote for A in this round. Similarly, Councillor Right has a dominant strategy to vote for D. Neither of these councillors misrepresent their preferences or vote strategically in either round. Councillor Center, however, has a dominant strategy to vote for D here even though she strictly prefers A to D. As the preceding discussion suggested, she has a strong incentive to misrepresent her preferences in the first round of voting; and she is the only one who votes strategically. Center's behavior changes the winner of the election from G (the winner without strategic voting) to D.

Remember that the chair, Councillor Left, set the agenda in the hope of having her most-preferred alternative chosen. Instead, her *least*-preferred alternative has prevailed. It appears that the power to set the agenda may not be so beneficial after all. But Councillor Left should anticipate the strategic behavior. Then she can choose the agenda so as to take advantage of her understanding of games of strategy. In fact, if she sets D against G in the first round and then the winner against A, the Nash equilibrium outcome is G, the chair's most-preferred outcome. With that agenda, Right misrepresents her preferences in the first round to vote for G over D to prevent A, her least-preferred outcome, from winning. You should verify that this is Councillor Left's best agenda-setting strategy. In the full voting game where setting the agenda is considered an initial, prevoting round, we should expect to see the Generous welfare policy adopted when Councillor Left is chair.

We can also see an interesting pattern emerge when we look more closely at voting behavior in the strategic version of the election. There are pairs of councillors who vote "together" (the same as one another) in both rounds. Under the original agenda, Right and Center vote together in both rounds, and in the suggested alternative (D versus G in the first round), Right and Left vote together in both rounds. In other words, a sort of long-lasting coalition has formed between two councillors in each case.

Strategic voting of this type appears to have taken place in Congress on more than one occasion. One example was a federal school-construction-funding bill considered in 1956.[15] Before being brought to a vote against the status quo of no funding, the bill was amended in the House of Representatives to require that aid be offered only to states with no racially segregated schools. Under the parliamentary voting rules of Congress, a vote on whether to accept the so-called Powell Amendment was taken first, with the winning version of the bill considered afterward. Political scientists who have studied the history of this bill argue that opponents of school funding strategically misrepresented their preferences regarding the amendment to defeat the original bill. A key group of Representatives voted for the amendment but then joined opponents of racial integration in voting against the full bill in the final vote; the bill was defeated. Voting records of this group indicate that many of them had voted against racial integration matters in other circumstances, implying that their vote for integration in this case was merely an instance of strategic voting and not an indication of their true feelings regarding school integration.

C. Strategic Voting with Incomplete Information

The preceding analysis showed that sometimes committee members have incentives to vote strategically to prevent their least-preferred alternative from winning an election. Our example assumed that the council members knew the possible preference orderings and how many other councillors had those preferences. Now suppose information is incomplete; each council member knows the possible preference orderings, her own actual ordering, and the probabilities that each of the others have a particular ordering, but not the actual distribution of the different preference orderings among the other councillors. In this situation, each councillor's strategy needs to be conditioned on her beliefs about that distribution and on her beliefs about how truthful other voters will be.[16]

For an example, suppose that we still have a three-member council considering the three alternative welfare policies described earlier according to the

[15] A more complete analysis of the case can be found in Riker, *Liberalism Against Populism*, pp. 152–57.

[16] This result can be found in P. Ordeshook and T. Palfrey, "Agendas, Strategic Voting, and Signaling with Incomplete Information," *American Journal of Political Science*, vol. 32, no. 2 (May 1988), pp. 441–66. The structure of the example to follow is based on Ordeshook and Palfrey's analysis.

(original) agenda set by Councillor Left; that is, the council considers policies A and D in the first round with the winner facing G in the second round. We assume that there are still three different possible preference orderings, as illustrated in Figure 15.1, and that the councillors know that these orderings are the only possibilities. The difference is that no one knows for sure exactly how many councillors have each set of preferences. Rather, each councillor knows her own type, and she knows that there is some positive probability of observing each type of voter (Left, Center, or Right), with the probabilities p_L, p_C, and p_R summing to 1.

We saw earlier that all three councillors vote truthfully in the last round of balloting. We also saw that Left- and Right-type councillors vote truthfully in the first round as well. This result remains true in the incomplete information case. Right-type voters prefer to see D win the first-round election; given this preference, Right always does at least as well by voting for D over A (if both other councillors have voted the same way) and sometimes does better by voting this way (if the other two votes split between D and A). Similarly, Left-type voters prefer to see A survive to vie against G in round two; these voters always do at least as well as otherwise—and sometimes do better—by voting for A over D.

At issue then is only the behavior of the Center-type voters. Because they do not know the types of the other councillors and because they have an incentive to vote strategically for some preference distributions—specifically the case in which it is known for certain that there is one voter of each type—their behavior will depend on the probabilities that the various voter types may occur within the council. We consider here one of two polar cases in which a Center-type voter believes that other Center types will vote truthfully, and we look for a symmetric, pure-strategy Nash equilibrium. The case in which she believes that other Center types will vote strategically is taken up in the exercises.

To make outcome comparisons possible, we specify payoffs for the Center-type voter associated with the possible winning policies. Center-type preferences are A $>$ D $>$ G. Suppose that, if A wins, Center types receive a payoff of 1 and, if G wins, Center types receive a payoff of 0. If D wins, Center types receive some intermediate-level payoff, call it u, where $0 < u < 1$.

Now suppose our Center-type councillor must decide how to vote in the first round (A versus D) in an election in which she believes that both other voters vote truthfully, regardless of their type. If both voters choose either A or D, then Center's vote is immaterial to the final outcome; she is indifferent between A and D. If the other two voters split their votes, however, then Center can influence the election outcome. Her problem is that she needs to decide whether to vote truthfully herself.

If the other two voters split between A and D and if both are voting truthfully, then the vote for D must have come from a Right-type voter. But the vote for A could have come from *either* a Left type *or* a (truthful) Center type. If the A vote came from a Left-type voter, then Center knows that there is one voter of each type. If she votes truthfully for A in this situation, A will win the first round but lose to G in the end; Center's payoff will be 0. If Center votes strategically for D, D beats A and G, and Center's payoff is u. In contrast, if the A vote came from a Center-type voter, then Center knows there are two Center types and a Right type but no Left type on the council. In this case, a truthful vote for A helps A win the first round, and then A also beats G by a vote of 2 to 1 in round two; Center gets her highest payoff of 1. If Center were to vote strategically for D, D would win both rounds again and Center would get u.

To determine Center's optimal strategy, we need to compare her expected payoff from truthful voting with her expected payoff from strategic voting. With a truthful vote for A, Center's payoff depends on how likely it is that the other A vote comes from a Left type or a Center type. Those probabilities are straightforward to calculate. The probability that the other A vote comes from a Left type is just the probability of a Left type being one of the remaining voters, or $p_L/(p_L + p_C)$; similarly, the probability that the A vote comes from a Center type is $p_C/(p_L + p_C)$. Then Center's payoffs from truthful voting are 0 with probability $p_L/(p_L + p_C)$ and 1 with probability $p_C/(p_L + p_C)$, so the expected payoff is $p_C/(p_L + p_C)$. With a strategic vote for D, D wins regardless of the identity of the third voter—D wins with certainty—and so Center's expected payoff is just u. Center's final decision is to vote truthfully as long as $p_C/(p_L + p_C) > u$.

Note that Center's decision-making condition is an intuitively reasonable one. If the probability of there being more Center-type voters is large or relatively larger than the probability of having a Left-type voter, then the Center types vote truthfully. Voting strategically is useful to Center only when she is the only voter of her type on the council.

We add two additional comments on the existence of imperfect information and its implications for strategic behavior. First, if the number of councillors, n, is larger than three but odd, then the expected payoff to a Center type from voting strategically remains equal to u, and the expected payoff from voting truthfully is $[p_C/(p_L + p_C)]^{(n-1)/2}$.[17] Thus, a Center type should vote truthfully only when $[p_C/(p_L + p_C)]^{(n-1)/2} > u$. Because $p_C/(p_L + p_C) < 1$ and $u > 0$, this inequality will *never* hold for large enough values of n. This result tells us that a symmetric truthful-voting equilibrium can never persist in a large enough council! Second, imperfect information about the preferences of other voters yields additional

[17] A Center type can affect the election outcome only if all other votes are split evenly between A and D. Thus, there must be exactly $(n - 1)/2$ Right-type voters choosing D in the first round and $(n - 1)/2$ other voters choosing A. If those A voters are Left types, then A won't win the second-round election, and Center will get 0 payoff. For Center to get a payoff of 1, it must be true that all of the other A voters are Center types. The probability of this occurring is $[p_C/(p_L + p_C)]^{(n-1)/2}$; then Center's expected payoff from voting truthfully is as stated. See Ordeshook and Palfrey, p. 455.

scope for strategic behavior. With agendas that include more than two rounds, voters can use their early-round votes to signal their types. The extra rounds give other voters the opportunity to update their prior beliefs about the probabilities p_C, p_L, and p_R and a chance to act on that information. With only two rounds of pairwise votes, there is no time to use any information gained during round one, because truthful voting is a dominant strategy for all voters in the final round.

D. Scope for Manipulability

The extent to which a voting procedure is susceptible to strategic misrepresentation of preferences, or strategic manipulability by voters of the types illustrated above, is another topic that has generated considerable interest among voting theorists. Arrow does not require nonmanipulability in his theorem, but the literature has considered how such a requirement would relate to Arrow's conditions. Similarly, theorists have considered the scope for manipulability in various procedures, producing rankings of voting methods.

The economist William Vickrey, perhaps better known for his work on auctions (see Chapter 16), did some of the earliest work considering strategic behavior of voters. He pointed out that procedures satisfying Arrow's IIA assumption were most immune to strategic manipulation. He also set out several conditions under which strategic behavior is more likely to be attempted and be successful. In particular, he noted that situations with smaller numbers of informed voters and smaller sets of available alternatives may be most susceptible to manipulation, given a voting method that is itself manipulable. This result means, however, that weakening the IIA assumption to help voting procedures satisfy Arrow's conditions makes way for more manipulable procedures. In particular, Saari's intensity ranking version of IIA (called IBI), mentioned in Section 3.C, may allow more procedures to satisfy a modified version of Arrow's theorem but may simultaneously allow more manipulable procedures to do so.

Like Arrow's general result on the impossibility of preference aggregation, the general result on manipulability is a negative one. Specifically, the **Gibbard–Satterthwaite theorem** shows that if there are three or more alternatives to consider, the only voting procedure that prevents strategic voting is dictatorship: one voter is assigned the role of dictator, and her preferences determine the election outcome.[18] Combining the Gibbard–Satterthwaite outcome with Vickrey's discussion of IIA may help the reader understand why Arrow's theorem is often reduced to a consideration of which procedures can simultaneously satisfy nondictatorship and IIA.

[18] For the theoretical details on this result, see A. Gibbard, "Manipulation of Voting Schemes: A General Result," *Econometrica*, vol. 41, no. 4 (July 1973), pp. 587–601, and M. A. Satterthwaite, "Strategy-Proofness and Arrow's Conditions," *Journal of Economic Theory*, vol. 10 (1975), pp. 187–217. The theorem carries both their names because each proved the result independent of the other.

Finally, some theorists have argued that voting systems should be evaluated not on their ability to satisfy Arrow's conditions but on their tendency to encourage manipulation. The relative manipulability of a voting system can be determined by the amount of information about the preferences of other voters that is required by voters to manipulate an election successfully. Some research based on this criterion suggests that of the procedures so far discussed, plurality rule is the most manipulable (that is, requires the least information). In decreasing order of manipulability are approval voting, the Borda count, the amendment procedure, majority rule, and the Hare procedure (single transferable vote).[19]

It is important to note that the classification of procedures by level of manipulability depends only on the amount of information necessary to manipulate a voting system and is not based on the ease of putting such information to good use or whether manipulation is most easily achieved by individual voters or groups. In practice, the manipulation of plurality rule by *individual* voters is quite difficult.

5 THE MEDIAN VOTER THEOREM

All of the preceding sections have focused on the behavior, strategic and otherwise, of voters in multiple alternative elections. However, strategic analysis can also be applied to *candidate* behavior in such elections. Given a particular distribution of voters and voter preferences, candidates will, for instance, need to determine optimal strategies in building their political platforms. When there are just two candidates in an election, when voters are distributed in a "reasonable" way along the political spectrum, and when each voter has "reasonably" consistent (meaning singled-peaked) preferences, the **median voter theorem** tells us that both candidates will position themselves on the political spectrum at the same place as the median voter. The **median voter** is the "middle" voter in that distribution—more precisely, the one at the 50th percentile.

The full game here has two stages. In the first stage, candidates choose their locations on the political spectrum. In the second stage, voters elect one of the candidates. The general second-stage game is open to all of the varieties of strategic misrepresentation of preferences discussed earlier. Hence we have reduced the choice of candidates to two for our analysis to prevent such behavior from arising in equilibrium. With only two candidates, second-stage votes will directly correspond to voter preferences, and the first-stage location

[19] H. Nurmi's classification can be found in his *Comparing Voting Systems* (Norwell, Mass.: D. Reidel, 1987).

decision of the candidates remains the only truly interesting part of the larger game. It is in that stage that the median voter theorem defines Nash equilibrium behavior.

A. Discrete Political Spectrum

Let us first consider a population of 90 million voters, each of whom has a preferred position on a five-point political spectrum: Far Left (FL), Left (L), Center (C), Right (R), and Far Right (FR). We suppose that these voters are spread symmetrically around the center of the political spectrum. The **discrete distribution** of their locations is shown by a **histogram**, or bar chart, in Figure 15.7. The height of each bar indicates the number of voters located at that position. In this example, we have supposed that, of the 90 million voters, 40 million are Left, 20 million are Far Right, and 10 million each are Far Left, Center, and Right.

Voters will vote for the candidate who publicly identifies herself as being closer to their own position on the spectrum in an election. If both candidates are politically equidistant from a group of like-minded voters, each voter flips a coin to decide which candidate to choose; this process gives each candidate one-half of the voters in that group.

Now suppose there is an upcoming presidential election between a former First Lady (Claudia) and a former First Lady hopeful (Dolores), each now running for office on her own.[20] Under the configuration of voters illustrated

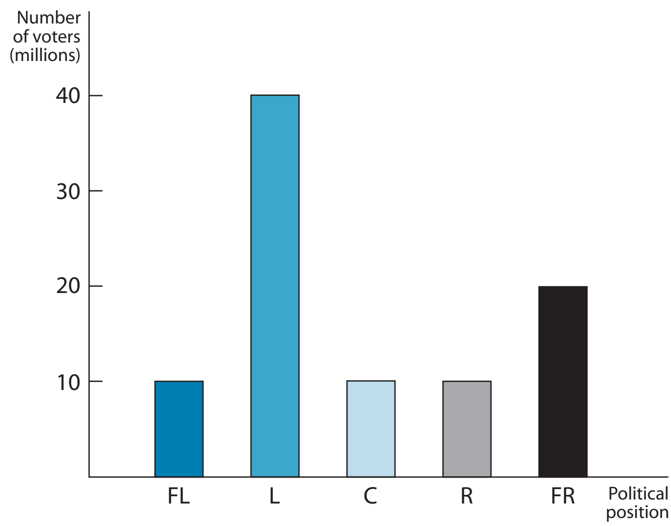

FIGURE 15.7 Discrete Distribution of Voters

[20] Any resemblance between our hypothetical candidates and actual past or possible future candidates in the United States is not meant to imply an analysis or prediction of their performances relative to the Nash equilibrium. Nor is our distribution of voters meant to typify U.S. voter preferences.

in Figure 15.7, we can construct a payoff table for the two candidates showing the number of votes that each can expect to receive under all of the different combinations of political platform choices. This five-by-five table is shown in Figure 15.8, with totals denoted in millions of votes. The candidates will choose their optimal location strategies to maximize the number of votes that they receive (and thus increase the chances of winning).[21]

Here is how the votes are allocated. When both candidates choose the *same* position (the five cells along the top-left to bottom-right diagonal of the table), each candidate gets exactly one-half of the votes; because all voters are equidistant from each candidate, all of them flip coins to decide their choices, and each candidate garners 45 million votes. When the two candidates choose *different* positions, the more-left candidate gets all the votes at or to the left of her position while the more-right candidate gets all the votes at or to the right of her position. In addition, each candidate gets the votes in central positions closer to her than to her rival, and the two of them split the votes from any voters in a central position equidistant between them. Thus, if Claudia locates herself at L while Dolores locates herself at FR, Claudia gets the 40 million votes at L, the 10 million at FL, *and* the 10 million at C (because C is closer to L than to FR). Dolores gets the 20 million votes at FR and the 10 million at R (because R is closer to FR than to L). The payoff is (60, 30). Similar calculations determine the outcomes in the rest of the table.

The table in Figure 15.8 is large, but the game can be solved very quickly. We begin with the now familiar search for dominant, or dominated, strategies for the two players. Immediately we see that for Claudia, FL is dominated by L and FR is dominated by R. For Dolores, too, her FL is dominated by L and FR by R. With these extreme strategies eliminated, for each candidate her R is dominated

		DOLORES				
		FL	L	C	R	FR
CLAUDIA	FL	45, 45	10, 80	30, 60	50, 40	55, 35
	L	80, 10	45, 45	50, 40	55, 35	60, 30
	C	60, 30	40, 50	45, 45	60, 30	65, 25
	R	40, 50	35, 55	30, 60	45, 45	70, 20
	FR	35, 55	30, 60	25, 65	20, 70	45, 45

FIGURE 15.8 Payoff Table for Candidates' Positioning Game

[21] To keep the analysis simple, we ignore the complications created by the electoral college and suppose that only the popular vote matters.

by C. With the two R strategies gone, C is dominated by L for each candidate. The only remaining cell in the table is (L, L); this is the Nash equilibrium.

We now note three important characteristics of the equilibrium in the candidate-location game. First, both candidates locate at the *same* position in equilibrium. This illustrates the **principle of minimum differentiation**, a general result in all two-player games of locational competition, whether it be political platform choice by presidential candidates, hotdog-cart location choices by street vendors, or product feature choices by electronics manufacturing firms.[22] When the persons who vote for or buy from you can be arranged on a well-defined spectrum of preferences, you do best by looking as much like your rival as possible. This explains a diverse collection of behaviors on the part of political candidates and businesses. It may help you understand, for example, why there is never just one gas station at a heavily traveled intersection or why all brands of four-door sedans (or minivans or sport utility vehicles) seem to look the same even though every brand claims to be coming out continually with a "new" look.

Second and perhaps most crucial, both candidates locate at the position of the median voter in the population. In our example, with a total of 90 million voters, the median voter is number 45 million from each end. The numbers within one location can be assigned arbitrarily, but the location of the median voter is clear; here, the median voter is located at the L position on the political spectrum. So that is where both candidates locate themselves, which is the result predicted by the median voter theorem.

Third, observe that the location of the median voter need not coincide with the geometric center of the spectrum. The two will coincide if the distribution of voters is symmetric, but the median voter can be to the left of the geometric center if the distribution is skewed to the left (as is true in Figure 15.7) and to the right if the distribution is skewed to the right. This helps explain why state political candidates in Massachusetts, for example, *all* tend to be more liberal than candidates for similar positions in Texas or South Carolina.

The median voter theorem can be expressed in different ways. One version states simply that the position of the median voter is the equilibrium-location position of the candidates in a two-candidate election. Another version says that the position that the median voter most prefers will be the Condorcet winner; this position will defeat every other position in a pairwise contest. For example, if M is this median position and L is any position to the left of M, then M will get all the votes of people who most prefer a position at or to the right of M, plus some to the left of M but closer to M than to L. Thus, M will get more than 50% of the votes. The two versions amount to the same thing because, in a two-candidate election, both seeking to win a majority will adopt

[22] Economists learn this result within the context of Hotelling's model of spatial location. See Harold Hotelling, "Stability in Competition," *Economic Journal*, vol. 39, no. 1 (March 1929), pp. 41–57.

the Condorcet-winner position. These interpretations are identical. In addition, to guarantee that the result holds for a particular population of voters, the theorem (in either form) requires that each voter's preferences be "reasonable," as suggested earlier. *Reasonable* here means "single peaked," as in Black's condition described in Section 3.A and Figure 15.4. Each voter has a unique, most-preferred position on the political spectrum, and her utility (or payoff) decreases away from that position in either direction.[23] In actual U.S. presidential elections, the theorem is borne out by the tendency for the main candidates to make very similar promises to the electorate.

B. Continuous Political Spectrum

The median voter theorem can also be proved for a continuous distribution of political positions. Rather than having five, three, or any finite number of positions from which to choose, a **continuous distribution** assumes there are effectively an infinite number of political positions. These political positions are then associated with locations along the real number line between 0 and 1.[24] Voters are still distributed along the political spectrum as before, but because the distribution is now continuous rather than discrete, we use a voter **distribution function** rather than a histogram to illustrate voter locations. Two common functions—the **uniform distribution** and the (symmetric) **normal distribution**—are illustrated in Figure 15.9.[25] The area under each curve represents the total number of votes

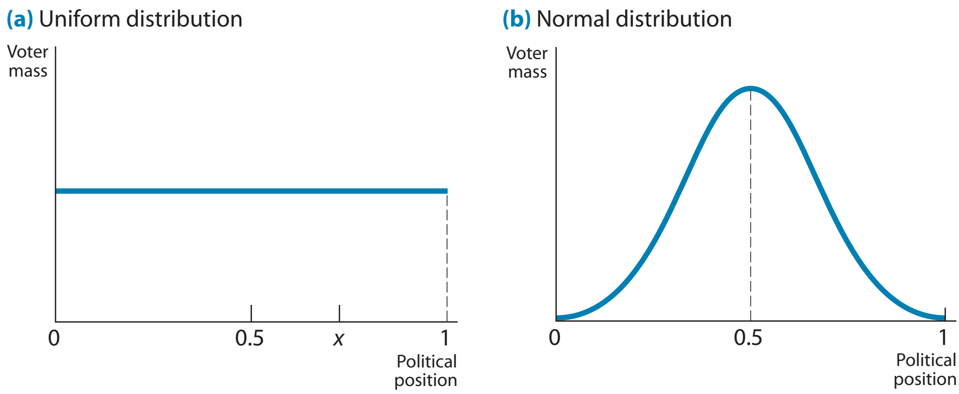

FIGURE 15.9 Continuous Voter Distributions

[23] However, the distribution of voters' ideal points along the political spectrum does not have to be single peaked, as indeed the histogram in Figure 15.7 is not—there are two peaks at L and FR.
[24] This construction is the same one used in Chapters 11 and 12 for analyzing large populations of individual members.
[25] We do not delve deeply into the mechanics underlying distribution theory or the integral calculus required to calculate the exact proportion of the voting population lying to the left or right of any particular position on the continuous political spectrum. Here we present only enough information to convince you that the median voter theorem continues to hold in the continuous case.

available; at any given point along the interval from 0 to 1, such as x in Figure 15.9a, the number of votes up to that point is determined by finding the area under the distribution function from 0 to x. It should be clear that the median voter in each of these distributions is located at the center of the spectrum, at position 0.5.

It is not feasible to construct a payoff table for our two candidates in the continuous-spectrum case; such tables must necessarily be finitely dimensioned and thus cannot accommodate an infinite number of possible strategies for players. We can, however, solve the game by applying the same strategic logic that we used for the discrete (finite) case discussed in Section 5.A.

Consider the options of Claudia and Dolores as they contemplate the possible political positions open to them. Each knows that she must find her Nash equilibrium strategy—her best response to the equilibrium strategy of her rival. We can define a set of strategies that are best responses quite easily in this game, even though the complete set of possible strategies is impossible to delineate.

Suppose Dolores locates at a random position on the political spectrum, such as x in Figure 15.9a. Claudia can then calculate how the votes will be split for all possible positions that she might choose. If she chooses a position to the left of x, she gets all the votes to her left and half of the votes lying between her position and Dolores's. If she locates to the right of x, she gets all the votes to her right and half of the votes lying between her position and x. Finally, if she, too, locates at x, she and Dolores split the votes 50–50. These three possibilities effectively summarize all of Claudia's location choices, given that Dolores has chosen to locate at x.

But which of the response strategies just outlined is Claudia's "best" response? The answer depends on the location of x relative to the median voter. If x is to the right of the median, then Claudia knows that her best response will be to maximize the number of votes that she gains, which she can do by locating an infinitely small bit to the left of x.[26] In that case, she effectively gets all the votes from 0 to x, and Dolores gets those from x to 1. When x is to the right of the median, as in Figure 15.9a, then the number of voters represented by the area under the distribution curve from 0 to x is by definition larger than the number of voters from x to 1, so Claudia would win the election. Similarly, if x is to the left of the median, Claudia's best response will be to locate an infinitely small bit to the right of x and thus gain all the votes from x to 1. When x is exactly at the median, Claudia does best by also choosing to locate at x.

[26] Such a location, infinitesimally removed from x to the left, is feasible in the continuous case. In our discrete example, candidates had to locate at exactly the same position.

The best-response strategies for Dolores are constructed exactly the same way and, given the location of her rival, are exactly the same as those described for Claudia. Graphically, these best-response curves lie just above and below the 45° line up to the position of the median voter, at which point they lie exactly on the 45° line. (Claudia's best response to Dolores's location at that of the median voter is to locate in the same place; the same is true in reverse for Dolores.) Beyond the position of the median voter, the best-response curves switch sides of the 45° line.

We now have complete descriptions of the best-response strategies for both candidates. The Nash equilibrium occurs at the intersection of the best-response curves; this intersection lies at the position of the median voter. You can think this through intuitively by picking any starting location for one of the candidates and applying the best-response strategies over and over until each candidate is located at a position that represents her best response to the position chosen by her rival. If Dolores were contemplating locating at x in Figure 15.9a, Claudia would want to locate just to the left of x, but then Dolores would want to locate just to the left of that, and so on. Only when the two candidates locate exactly at the median of the distribution (whether the distribution is uniform or normal or some other kind) do they find that their decisions are best responses to each other. Again, we see that the Nash equilibrium is for both candidates to locate at the position of the median voter.

More complex mathematics is needed to prove the continuous version of the median voter theorem to the satisfaction of a true mathematician. For our purposes, however, the discussion given here should convince you of the validity of the theorem in both its discrete and continuous forms. The most important limitation of the median voter theorem is that it applies when there is just one issue, or on a one-dimensional spectrum of political differences. If there are two or more dimensions—for example, if being conservative versus liberal on social issues does not coincide with being conservative versus liberal on economic issues—then the population is spread out in a two-dimensional "issue space" and the median voter theorem no longer holds. The preferences of every individual voter can be single peaked, in the sense that the individual voter has a most-preferred point and her payoff value drops away from this point in all directions, like the height going away from the peak of a hill. But we cannot identify a median voter in two dimensions, such that exactly the same number of voters have their most-preferred point to the one side of the median voter position as to the other side. In two dimensions, there is no unique sense of side, and the numbers of voters to the two sides can vary, depending on just how we define "side."

SUMMARY

Elections can be held with the use of a variety of different voting procedures that alter the order in which issues are considered or the manner in which votes are tallied. Voting procedures are classified as *binary*, *plurative*, or *mixed methods*. Binary methods include *majority rule*, as well as *pairwise* procedures such as the *Condorcet method* and the *amendment procedure*. *Positional methods* such as *plurality rule* and the *Borda count*, as well as *approval voting*, are plurative methods. And *majority runoffs*, *instant runoffs*, and *proportional representation* are mixed methods.

Voting paradoxes (such as the *Condorcet*, the *agenda*, and the *reversal paradox*) show how counterintuitive results can arise owing to difficulties associated with aggregating preferences or to small changes in the list of issues being considered. Another paradoxical result is that outcomes in any given election under a given set of voter preferences can change, depending on the voting procedure used. Certain principles for evaluating voting methods can be described, although Arrow's *impossibility theorem* shows that no one system satisfies all of the criteria at the same time. Researchers in a broad range of fields have considered alternatives to the principles that Arrow identified.

Voters have scope for strategic behavior in the game that chooses the voting procedure or in an election itself through the *misrepresentation of their own preferences*. Voters may strategically misrepresent preferences to achieve their most-preferred or to avoid their least-preferred outcome. In the presence of imperfect information, voters may decide whether to vote strategically on the basis of their beliefs about others' behavior and their knowledge of the distribution of preferences.

Candidates also may behave strategically in building a political platform. A general result known as the *median voter theorem* shows that in elections with only two candidates, both locate at the preference position of the *median voter*. This result holds when voters are distributed along the preference spectrum either *discretely* or *continuously*.

KEY TERMS

agenda paradox (597)
amendment procedure (591)
antiplurality method (592)
approval voting (592)
binary method (591)
Black's condition (601)
Borda count (592)
Condorcet method (591)
Condorcet paradox (595)
Condorcet terms (603)
Condorcet winner (591)
continuous distribution (617)

SOLVED EXERCISES

S1. Consider a vote being taken by three roommates, A, B, and C, who share a triple dorm room. They are trying to decide which of three elective courses to take together this term. (Each roommate has a different major and is taking required courses in her major for the rest of her courses.) Their choices are Philosophy, Geology, and Sociology, and their preferences for the three courses are as shown here:

A	B	C
Philosophy	Sociology	Geology
Geology	Philosophy	Sociology
Sociology	Geology	Philosophy

The roommates have decided to have a two-round vote and will draw straws to determine who sets the agenda. Suppose A sets the agenda and wants the Philosophy course to be chosen. How should she set the agenda to achieve this outcome if she knows that everyone will vote truthfully in all rounds? What agenda should she use if she knows that they will all vote strategically?

S2. Suppose that voters 1 through 4 are being asked to consider three different candidates—A, B, and C—in a Borda-count election. Their preferences are:

1	2	3	4
A	A	B	C
B	B	C	B
C	C	A	A

Assume that voters will cast their votes truthfully (no strategic voting). Find a Borda weighting system—a number of points to be allotted to the first, second, and third preferences—in which candidate A wins.

S3. Consider a group of 50 residents attending a town meeting in Massachusetts. They must choose one of three proposals for dealing with town garbage. Proposal 1 asks the town to provide garbage collection as one of its services; Proposal 2 calls for the town to hire a private garbage collector to provide collection services; and Proposal 3 calls for residents to be responsible for their own garbage. There are three types of voters. The first type prefers Proposal 1 to Proposal 2 and Proposal 2 to Proposal 3; there are 20 of these voters. The second type prefers Proposal 2 to Proposal 3 and Proposal 3 to Proposal 1; there are 15 of these voters. The third type prefers Proposal 3 to Proposal 1 and Proposal 1 to Proposal 2; there are 15 of them.

(a) Under a plurality voting system, which proposal wins?

(b) Suppose voting proceeds with the use of a Borda count in which voters list the proposals, in order of preference, on their ballots. The proposal listed first (or at the top) on a ballot gets three points; the proposal listed second gets two points; and the proposal listed last gets one point. In this situation, with no strategic voting, how many points does each proposal gain? Which proposal wins?

(c) What strategy can the second and third types of voters use to alter the outcome of the Borda-count vote in part (b) to one that both types prefer? If they use this strategy, how many points does each proposal get, and which wins?

S4. During the Cuban missile crisis, serious differences of opinion arose within the ExComm group advising President John F. Kennedy, which we summarize here. There were three options: Soft (a blockade), Medium (a limited air strike), and Hard (a massive air strike or invasion). There were also three groups in ExComm. The civilian doves ranked the alternatives

Soft best, Medium next, and Hard last. The civilian hawks preferred Medium best, Hard next, and Soft last. The military preferred Hard best, but they felt "so strongly about the dangers inherent in the limited strike that they would prefer taking no military action rather than to take that limited strike." [Ernest R. May and Philip D. Zelikow, eds., *The Kennedy Tapes: Inside the White House During the Cuban Missile Crisis* (Cambridge, Mass.: Harvard University Press, 1997), p. 97.] In other words, they ranked Soft second and Medium last. Each group constituted about one-third of ExComm, and so any two of the groups would form a majority.

(a) If the matter were to be decided by a majority vote in ExComm and the members voted sincerely, which alternative, if any, would win?

(b) What outcome would arise if members voted strategically? What outcome would arise if one group had agenda-setting power? (Model your discussion in these two cases after the analysis found in Sections 2.B and 4.B.)

S5. In his book *A Mathematician Reads the Newspaper*, John Allen Paulos gives the following caricature based on the 1992 Democratic presidential primary caucuses. There are five candidates: Jerry Brown, Bill Clinton, Tom Harkin, Bob Kerrey, and Paul Tsongas. There are 55 voters, with different preference orderings concerning the candidates. There are six different orderings, which we label I through VI. The preference orderings (1 for best to 5 for worst), along with the numbers of voters with each ordering, are shown in the following table (the candidates are identified by the first letters of their last names)[27]:

		GROUPS AND THEIR SIZES					
		I 18	II 12	III 10	IV 9	V 4	VI 2
RANKING	1	T	C	B	K	H	H
	2	K	H	C	B	C	B
	3	H	K	H	H	K	K
	4	B	B	K	C	B	C
	5	C	T	T	T	T	T

(a) First, suppose that all voters vote sincerely. Consider the outcomes of each of several different election rules. Show each of the following outcomes: (i) Under the plurality method (the one with the most

[27] John Allen Paulos, *A Mathematician Reads the Newspaper* (New York: Basic Books, 1995), pp. 104–106.

first preferences), Tsongas wins. (ii) Under the runoff method (the top two first preferences go into a second round), Clinton wins. (iii) Under the elimination method (at each round, the one with the fewest first preferences in that round is eliminated, and the rest go into the next round), Brown wins. (iv) Under the Borda-count method (5 points for first preference, 4 for second, and so on; the candidate with the most points wins), Kerrey wins. (v) Under the Condorcet method (pairwise comparisons), Harkin wins.

(b) Suppose that you are a Brown, Kerrey, or Harkin supporter. Under the plurality method, you would get your worst outcome. Can you benefit by voting strategically? If so, how?

(c) Are there opportunities for strategic voting under each of the other methods as well? If so, explain who benefits from voting strategically and how they can do so.

S6. As mentioned in the chapter, some localities (such as San Francisco) have replaced runoff elections and even primaries with instant runoff voting to save time and money. Most jurisdictions have implemented a two-stage system in which if a candidate fails to receive a majority of votes in the first round, a second runoff election is held weeks later between the two candidates who earned the most votes.

For instance, France employs a two-stage system for its presidential elections. No primaries are held. Instead, all candidates from all parties are on the ballot in the first round, which usually guarantees a second round, since it is very difficult for a single candidate to earn a majority of votes among such a large field. Although a runoff in the French presidential election is always expected, it doesn't mean that French elections are not without the occasional surprise. In 2002, the country was shocked when the right-wing candidate Jean-Marie Le Pen beat the socialist Lionel Jospin to take second place and thus advance to the runoff election against the first-round winner (and incumbent) Jacques Chirac. It had been widely assumed that Jospin would take second, setting up a runoff between himself and Chirac.

Instant runoff voting can be explained in five steps:

1. Voters rank all candidates according to their preferences.
2. The votes are counted.
3. If a candidate has earned a majority of the votes, that candidate is the winner. If not, go to step 4.
4. Eliminate candidate(s) with the fewest votes. (Eliminate more than one candidate at the same time only if they tie for the fewest votes.)
5. Redistribute votes from eliminated candidates to the next-ranked choices on those ballots. Once this is done, return to step 2.

(a) Instant runoff voting is slowly gaining traction. It is used in a dozen cities in the United States and for state-wide judicial elections in North Carolina (as of 2013). Given the potential savings in money and time, it might be surprising that the institution isn't more widely adopted. Why might some oppose instant runoff voting? (Hint: Which candidates, parties, and interests benefit from the two-stage system that is currently in place?)

(b) What other concerns or criticisms might be raised about instant runoff voting?

S7. An election has three candidates and takes place under the plurality rule. There are numerous voters, spread along an ideological spectrum from left to right. Represent this spread by a horizontal straight line whose extreme points are 0 (left) and 1 (right). Voters are uniformly distributed along this spectrum; so the number of voters in any segment of the line is proportional to the length of that segment. Thus, a third of the voters are in the segment from 0 to 1/3, a quarter in the segment from 1/2 to 3/4, and so on. Each voter votes for the candidate whose declared position is closest to the voter's own position. The candidates have no ideological attachment and take up any position along the line, each seeking only to maximize her share of votes.

(a) Suppose you are one of the three candidates. The leftmost of the other two is at point x, and the rightmost is at the point $(1 - y)$, where $x + y < 1$ (so the rightmost candidate is a distance y from 1). Show that your best response is to take up the following positions under the given conditions:

(i) just slightly to the left of x if $x > y$ and $3x + y > 1$;

(ii) just slightly to the right of $(1 - y)$ if $y > x$ and $x + 3y > 1$; and

(iii) exactly halfway between the other candidates if $3x + y < 1$ and $x + 3y < 1$.

(b) In a graph with x and y along the axes, show the areas (the combination of x and y values) where each of the response rules [(i) to (iii) in part (a)] is best for you.

(c) From your analysis, what can you conclude about the Nash equilibrium of the game where the three candidates each choose positions?

UNSOLVED EXERCISES

U1. Repeat Exercise S1 for the situation in which B sets the agenda and wants to ensure that Sociology wins.

U2. Repeat Exercise S2 to find a Borda weighting system in which candidate B wins.

U3. Every year, college football's Heisman Trophy is awarded by means of a Borda-count voting system. Each voter submits first-, second-, and third-place votes, worth 3 points, 2 points, and 1 point, respectively. Thus, the Borda-count point scheme used may be called (3-2-1), where the first digit is the point value of a first-place vote, the second digit denotes the value of a second-place vote, and the third digit gives the point value of a third-place vote. In 2004, the vote totals for the top five under the Borda system were as follows:

Player	1st Place	2nd Place	3rd Place
Leinhart (USC)	267	211	102
Peterson (Oklahoma)	154	180	175
White (Oklahoma)	171	149	146
Smith (Utah)	98	112	117
Bush (USC)	118	80	83

(a) Compare the Borda point scores of Leinhart and Peterson. By what margin of Borda points did Leinhart win?

(b) It seems only fair that a point scheme should give a first-place vote at least as much weight as a second-place vote and a second-place vote at least as much weight as a third-place vote. That is, for a point scheme (x-y-z), we should have $x \geq y \geq z$. Given this "fairness" restriction, is there any point scheme under which Leinhart would have lost? If so, provide such a scheme. If not, explain why not.

(c) Even though White had more first-place votes than Peterson, Peterson had a higher Borda-count total. If first-place votes were weighted enough, White's edge in first-place votes could give him a higher Borda count. Assume that second-place votes are worth 2 points and third-place votes are worth 1 point, so that the point scheme is (x-2-1). What is the lowest integer value of x such that White gets a higher Borda count than Peterson?

(d) Suppose that the above vote data represent truthful voting. For simplicity, let's suppose that the election were a simple plurality vote instead of a Borda count. Note that Leinhart and Bush are both from USC, whereas Peterson and White are both from Oklahoma. Suppose that, due to Oklahoma loyalty, those voters who prefer White all have Peterson as their second choice. If these voters were to vote strategically in a plurality election, could they change the outcome of the election? Explain.

(e) Similarly, suppose that due to USC loyalty, those voters who prefer Bush all have Leinhart as their second choice. If all four voting groups (Leinhart, Peterson, White, Bush) were to vote strategically in a plurality election, who would be the winner of the Heisman Trophy?

(f) In 2004, there were 923 Heisman voters. Under the actual (3-2-1) system, what is the minimum integer number of first-place votes that it would have taken to guarantee victory (that is, without the help of any second- or third-place votes)? Note that a player's name may appear on a ballot only once.

U4. Olympic skaters complete two programs in their competition, one short and one long. In each program, the skaters are scored and then ranked by a panel of nine judges. The skaters' positions in the rankings are used to determine their final scores. A skater's ranking depends on the number of judges placing her first (or second or third); the skater judged to be best by the most judges is ranked number one, and so on. In the calculation of a skater's final score, the short program gets half the weight of the long program. That is, Final score = 0.5 (Rank in short program) + Rank in long program. The skater with the lowest final score wins the gold medal. In the event of a tie, the skater judged best in the long program by the most judges takes the gold. In the 2002 women's individual figure-skating competition in Salt Lake City, Michelle Kwan was in first place after the short program. She was followed by Irina Slutskaya, Sasha Cohen, and Sarah Hughes, who were in second, third, and fourth places, respectively. In the long program, the judges' cards for these four skaters were as follows:

		JUDGE NUMBER								
		1	2	3	4	5	6	7	8	9
KWAN	Points	11.3	11.5	11.7	11.5	11.4	11.5	11.4	11.5	11.4
	Rank	2	3	2	2	2	3	3	2	3
SLUTSKAYA	Points	11.3	11.7	11.8	11.6	11.4	11.7	11.5	11.4	11.5
	Rank	3	1	1	1	4	1	2	3	2
COHEN	Points	11.0	11.6	11.5	11.4	11.4	11.4	11.3	11.3	11.3
	Rank	4	2	4	3	3	4	4	4	4
HUGHES	Points	11.4	11.5	11.6	11.4	11.6	11.6	11.3	11.6	11.6
	Rank	1	4	3	4	1	2	1	1	1

(a) At the Olympics, Slutskaya skated last of the top skaters. Use the information from the judges' cards to determine the judges' long-program ranks for Kwan, Cohen, and Hughes *before* Slutskaya skated. Then, using the standings already given for the short program in conjunction with your calculated ranks for the long program, determine the final scores and standings among these three skaters *before* Slutskaya skated. (Note that Kwan's rank in the short program was 1, and so her partial score after the short program is 0.5.)

(b) Given your answer to part (a), what would have been the final outcome of the competition if the judges had ranked Slutskaya's long program above all three of the others?

(c) Use the judges' cards to determine the actual final scores for all four skaters *after* Slutskaya skated. Who won each medal?

(d) What important principle, of those identified by Arrow, does the Olympic figure-skating scoring system violate? Explain.

U5. The 2008 presidential nomination season saw 21 Republican primaries and caucuses on Super Tuesday—February 5, 2008. By that day—only a month after the Iowa caucus that began the process—more than half of the Republican contenders had dropped out of the race, leaving only four: John McCain, Mitt Romney, Mike Huckabee, and Ron Paul. McCain, Romney, and Huckabee had each previously won at least one state. McCain had beaten Romney in Florida the week before the big day, and at that point it looked like only the two of them stood a realistic chance of wining the nomination. In this primary season, as is typical for the Republican party, nearly every GOP contest (whether primary or caucus) was winner-take-all, so winning a given state would earn a candidate all of the delegates allotted to that state by the Republican National Committee.

The West Virginia caucus was the first contest to reach a conclusion on Super Tuesday, as the caucus took place in the afternoon, it was brief, and the state is in the eastern time zone. News of the result was available hours before the close of polls in many of the states voting that day.

The following problem is based on the results of that West Virginia caucus. As we might expect, the caucusers did not all share the same preferences over the candidates. Some favored McCain, whereas others liked Romney or Huckabee. The voters also certainly had varied preferences about whom they wanted to win if their favorite candidate did not. Simplifying substantially from reality (but based on the actual voting), assume that there were seven types of West Virginia caucus goers that day, whose prevalence and preferences were as follows:

	I (16%)	II (28%)	III (13%)	IV (21%)	V (12%)	VI (6%)	VII (4%)
1st	McCain	Romney	Romney	Huckabee	Huckabee	Paul	Paul
2nd	Romney	McCain	Huckabee	Romney	McCain	Romney	Huckabee
3rd	Huckabee	Huckabee	McCain	McCain	Romney	Huckabee	Romney
4th	Paul	Paul	Paul	Paul	Paul	McCain	McCain

At first, no one knew the distribution of preferences of those in attendance at the caucus, so everyone voted truthfully. Thus, Romney won a plurality of the votes in the first round with 41%.

After each round of this caucus, if no candidate wins a majority, the candidate with the least number of votes is dropped from consideration, and his or her supporters vote for one of the remaining candidates in the following rounds.

(a) What would the results of the second round have been under truthful (nonstrategic) voting for the remaining three candidates?

(b) If West Virginia had held pairwise votes among the four candidates, which one would have been the Condorcet winner with truthful voting?

(c) In reality, the results of the second round of the caucus were:

Huckabee:	52%
Romney:	47%
McCain:	1%

Given the preferences of the McCain voters, why might this have happened? (Hint: How would the outcome have been different if West Virginia had voted last on Super Tuesday?)

(d) After the fact, Romney's campaign cried foul and accused the McCain and Huckabee supporters of making a backroom deal (see Susan Davis, "Romney Cries Foul in W. Va. Loss," *Wall Street Journal*, February 5, 2008. Available at http://blogs.wsj.com/washwire/2008/02/05/huckabee-wins-first-super-tuesday-contest/?mod=WSJBlog). Should Romney's campaign have suspected collusion between the McCain and Huckabee camps in this case? Explain why or why not.

U6. Return to the discussion of instant runoff voting (IRV) in Exercise S6.

(a) Consider the following IRV ballots of five voters:

	Ana	Bernard	Cindy	Desmond	Elizabeth
1st	Jack	Jack	Kate	Locke	Locke
2nd	Kate	Kate	Locke	Kate	Jack
3rd	Locke	Locke	Jack	Jack	Kate

Which—if any—of the five voters have an incentive to vote strategically? If so, who and why? If not, explain why not.

(b) Consider the following table, which gives the IRV ballots of a small town of seven citizens voting on five policy proposals put forward by the mayor:

	Anderson	Brown	Clark	Davis	Evans	Foster	García
1st	V	V	W	W	X	Y	Z
2nd	W	X	V	X	Y	X	Y
3rd	X	W	Y	V	Z	Z	X
4th	Y	Y	X	Y	V	W	W
5th	Z	Z	Z	Z	W	V	V

Assuming that all candidates (or policies) that tie for the fewest votes are eliminated at the same time, under what conditions is an eventual majority winner guaranteed? Put another way, under what conditions might there not be an unambiguous majority winner? (Hint: How important is it for Evans, Foster, and García to fill out their ballots completely?) How will these conditions change if Harris moves into town and votes?

U7. Recall the three-member council considering three alternative welfare policies in Section 4.C. There, three councillors (Left, Center, and Right) considered policies A and D in a first-round vote, with the winner facing policy G in a second-round election. But no one knows for sure exactly how many councillors have each set of possible preferences. The possible preference orderings are shown in Figure 15.1. Each councillor knows her own type, and she knows the probabilities of observing each type of voter, p_L, p_C, and p_R (with $p_L + p_C + p_R = 1$). The behavior of the

Center-type voters in the first-round election is the only unknown in this situation and will depend on the probabilities that the various preference types occur. Suppose here that a Center-type voter believes (in contrast with the case considered in the chapter) that other Center types will vote strategically; suppose further that the Center-type's payoffs are as in Section 4.C: 1 if A wins, 0 if G wins, and $0 < u < 1$ if D wins.

(a) Under what configuration of the other two votes does the Center-type voter's first-round vote matter to the outcome of the election? Given her assumption about the behavior of other Center-type voters, how would she identify the source of the first-round votes?

(b) Following the analysis in Section 4.C, determine the expected payoff to the Center type when she votes truthfully. Compare this with her expected payoff when she votes strategically. What is the condition under which the Center type votes strategically?