

3

Determinants

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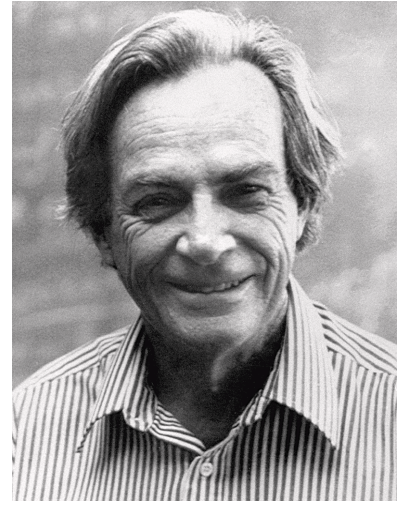
INTRODUCTORY EXAMPLE

Random Paths and Distortion

In his autobiographical book “Surely You’re Joking, Mr. Feynman,” the Nobel Prize–winning physicist Richard Feynman tells of observing ants in his Princeton graduate school apartment. He studied the ants’ behavior by providing paper ferries to sugar suspended on a string where the ants would not accidentally find it. When an ant would step onto a paper ferry, Feynman would transport the ant to the food and then back. After the ants learned to use the ferry, he relocated the return landing. The colony soon confused the outbound and return ferry landings, indicating that their “learning” consisted of creating and following trails. Feynman confirmed this conjecture by laying glass slides on the floor. Once the ants established trails on the glass slides, he rearranged the slides and therefore the trails on them. The ants followed the repositioned trails and Feynman could direct the ants where he wished.

Suppose Feynman had decided to conduct additional investigations using a globe built of wire mesh on which an ant must follow individual wires and choose between going left and right at each intersection. If several ants and an equal number of food sources are placed on the globe, how likely is it that each ant would find its own food source rather than encountering another ant’s trail and following it to a shared resource?¹

¹ The solution to the ant-path problem (and two other applications) can be found in a June 2005, *Mathematical Monthly* article by Arthur Benjamin and Naomi Cameron.



In order to record the actual routes of the ants and to communicate the results to others, it is convenient to use a rectangular map of the globe. There are many ways to create such maps. One simple way is to use the longitude and latitude on the globe as x and y coordinates on the map. As is the case with all maps, the result is not a faithful representation of the globe. Features near the “equator” look much the same on the globe and the map, but regions near the “poles” of the globe are distorted. Images of polar regions are much larger than the images of similar sized regions near the equator. To fit in with its surroundings on the map, the image of an ant near one of the poles should be larger than one near the equator. How much larger?

Surprisingly, both the ant-path and the area distortion problems are best answered through the use of the determinant, the subject of this chapter. Indeed, the determinant has so many uses that a summary of the applications known in the early 1900’s filled a four volume treatise by Thomas Muir. With changes in emphasis and the greatly increased sizes of the matrices used in modern applications, many uses that were important then are no longer critical today. Nevertheless, the determinant still plays an important role.

Beyond introducing the determinant in Section 3.1, this chapter presents two important ideas. Section 3.2 derives an invertibility criterion for a square matrix that plays a pivotal role in Chapter 5. Section 3.3 shows how the determinant measures the amount by which a linear transformation changes the area of a figure. When applied locally, this technique answers the question of a map's expansion rate near the poles. This idea plays a critical role in multivariable calculus in the form of the Jacobian.

3.1 INTRODUCTION TO DETERMINANTS

Recall from Section 2.2 that a 2×2 matrix is invertible if and only if its determinant is nonzero. To extend this useful fact to larger matrices, we need a definition for the determinant of an $n \times n$ matrix. We can discover the definition for the 3×3 case by watching what happens when an invertible 3×3 matrix A is row reduced.

Consider $A = [a_{ij}]$ with $a_{11} \neq 0$. If we multiply the second and third rows of A by a_{11} and then subtract appropriate multiples of the first row from the other two rows, we find that A is row equivalent to the following two matrices:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix} \quad (1)$$

Since A is invertible, either the (2, 2)-entry or the (3, 2)-entry on the right in (1) is nonzero. Let us suppose that the (2, 2)-entry is nonzero. (Otherwise, we can make a row interchange before proceeding.) Multiply row 3 by $a_{11}a_{22} - a_{12}a_{21}$, and then to the new row 3 add $-(a_{11}a_{32} - a_{12}a_{31})$ times row 2. This will show that

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad (2)$$

Since A is invertible, Δ must be nonzero. The converse is true, too, as we will see in Section 3.2. We call Δ in (2) the **determinant** of the 3×3 matrix A .

Recall that the determinant of a 2×2 matrix, $A = [a_{ij}]$, is the number

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

For a 1×1 matrix—say, $A = [a_{11}]$ —we define $\det A = a_{11}$. To generalize the definition of the determinant to larger matrices, we'll use 2×2 determinants to rewrite the 3×3 determinant Δ described above. Since the terms in Δ can be grouped as $(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$,

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

For brevity, write

$$\Delta = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13} \quad (3)$$

where A_{11} , A_{12} , and A_{13} are obtained from A by deleting the first row and one of the three columns. For any square matrix A , let A_{ij} denote the submatrix formed by deleting

the i th row and j th column of A . For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

so that

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

We can now give a *recursive* definition of a determinant. When $n = 3$, $\det A$ is defined using determinants of the 2×2 submatrices A_{1j} , as in (3) above. When $n = 4$, $\det A$ uses determinants of the 3×3 submatrices A_{1j} . In general, an $n \times n$ determinant is defined by determinants of $(n - 1) \times (n - 1)$ submatrices.

DEFINITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A . In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

SOLUTION Compute $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$:

$$\begin{aligned} \det A &= 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= 1(0 - 2) - 5(0 - 0) + 0(-4 - 0) = -2 \end{aligned}$$

Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets. Thus the calculation in Example 1 can be written as

$$\det A = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

To state the next theorem, it is convenient to write the definition of $\det A$ in a slightly different form. Given $A = [a_{ij}]$, the **(i, j) -cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij} \quad (4)$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

This formula is called a **cofactor expansion across the first row** of A . We omit the proof of the following fundamental theorem to avoid a lengthy digression.

THEOREM 1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

The plus or minus sign in the (i, j) -cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} itself. The factor $(-1)^{i+j}$ determines the following checkerboard pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

EXAMPLE 2 Use a cofactor expansion across the third row to compute $\det A$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

SOLUTION Compute

$$\begin{aligned} \det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31} \det A_{31} + (-1)^{3+2}a_{32} \det A_{32} + (-1)^{3+3}a_{33} \det A_{33} \\ &= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 = -2 \end{aligned}$$

Theorem 1 is helpful for computing the determinant of a matrix that contains many zeros. For example, if a row is mostly zeros, then the cofactor expansion across that row has many terms that are zero, and the cofactors in those terms need not be calculated. The same approach works with a column that contains many zeros.

EXAMPLE 3 Compute $\det A$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

SOLUTION The cofactor expansion down the first column of A has all terms equal to zero except the first. Thus

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} - 0 \cdot C_{21} + 0 \cdot C_{31} - 0 \cdot C_{41} + 0 \cdot C_{51}$$

Henceforth we will omit the zero terms in the cofactor expansion. Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there. We have

$$\det A = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

This 3×3 determinant was computed in Example 1 and found to equal -2 . Hence $\det A = 3 \cdot 2 \cdot (-2) = -12$. ■

The matrix in Example 3 was nearly triangular. The method in that example is easily adapted to prove the following theorem.

THEOREM 2

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A .

The strategy in Example 3 of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros! So the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

NUMERICAL NOTE

By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires over $n!$ multiplications, and $25!$ is approximately 1.5×10^{25} .

If a computer performs one trillion multiplications per second, it would have to run for over 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

Exercises 19–38 explore important properties of determinants, mostly for the 2×2 case. The results from Exercises 33–36 will be used in the next section to derive the analogous properties for $n \times n$ matrices.

PRACTICE PROBLEM

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$

$$7. \begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix} \qquad 8. \begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

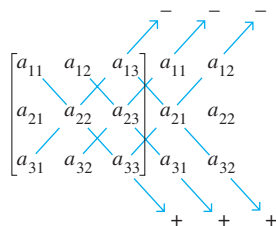
$$9. \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} \qquad 10. \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$$

$$11. \begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} \qquad 12. \begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$$

$$13. \begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

$$14. \begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:



Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. **Warning:** This trick does not generalize in any reasonable way to 4×4 or larger matrices.

$$15. \begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} \qquad 16. \begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$$

$$17. \begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} \qquad 18. \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

$$19. \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix} \qquad 20. \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

$$21. \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix}$$

$$22. \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$$

$$23. \begin{bmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}, \begin{bmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}$$

$$24. \begin{bmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{bmatrix}$$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

$$25. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix} \qquad 26. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

$$27. \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 28. \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$29. \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 30. \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$33. \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 34. \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$35. \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \qquad 36. \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 and 40, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

39. a. An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

b. The (i, j) -cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its i th row and j th column.

40. a. The cofactor expansion of $\det A$ down a column is the negative of the cofactor expansion along a row.

- b. The determinant of a triangular matrix is the sum of the entries on the main diagonal.
41. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by \mathbf{u} , \mathbf{v} , $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinant of $[\mathbf{u} \ \mathbf{v}]$. How do they compare? Replace the first entry of \mathbf{v} by an arbitrary number x , and repeat the problem. Draw a picture and explain what you find.
42. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where a, b, c are positive (for simplicity). Compute the area of the parallelogram determined by \mathbf{u} , \mathbf{v} , $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinants of the matrices $[\mathbf{u} \ \mathbf{v}]$ and $[\mathbf{v} \ \mathbf{u}]$. Draw a picture and explain what you find.
43. [M] Is it true that $\det(A + B) = \det A + \det B$? To find out, generate random 5×5 matrices A and B , and compute $\det(A + B) - \det A - \det B$. (Refer to Exercise 37 in Section 2.1.) Repeat the calculations for three other pairs of $n \times n$ matrices, for various values of n . Report your results.
44. [M] Is it true that $\det AB = (\det A)(\det B)$? Experiment with four pairs of random matrices as in Exercise 43, and make a conjecture.
45. [M] Construct a random 4×4 matrix A with integer entries between -9 and 9 , and compare $\det A$ with $\det A^T$, $\det(-A)$, $\det(2A)$, and $\det(10A)$. Repeat with two other random 4×4 integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 36 in Section 2.1.) Then check your conjectures with several random 5×5 and 6×6 integer matrices. Modify your conjectures, if necessary, and report your results.
46. [M] How is $\det A^{-1}$ related to $\det A$? Experiment with random $n \times n$ integer matrices for $n = 4, 5$, and 6 , and make a conjecture. *Note:* In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

SOLUTION TO PRACTICE PROBLEM

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a 3×3 matrix, which may be evaluated by an expansion down its first column.

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3} 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} \\ = 2 \cdot (-1)^{2+1} (-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

The $(-1)^{2+1}$ in the next-to-last calculation came from the $(2, 1)$ -position of the -5 in the 3×3 determinant.

3.2 PROPERTIES OF DETERMINANTS

The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19–24 in Section 3.1. The proof is at the end of this section.

THEOREM 3

Row Operations

Let A be a square matrix.

- If a multiple of one row of A is added to another row to produce a matrix B , then $\det B = \det A$.
- If two rows of A are interchanged to produce B , then $\det B = -\det A$.
- If one row of A is multiplied by k to produce B , then $\det B = k \cdot \det A$.

The following examples show how to use Theorem 3 to find determinants efficiently.

EXAMPLE 1 Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

SOLUTION The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$\det A = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15 \quad \blacksquare$$

A common use of Theorem 3(c) in hand calculations is to *factor out a common multiple of one row* of a matrix. For instance,

$$\begin{vmatrix} * & * & * \\ 5k & -2k & 3k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ 5 & -2 & 3 \\ * & * & * \end{vmatrix}$$

where the starred entries are unchanged. We use this step in the next example.

EXAMPLE 2 Compute $\det A$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

SOLUTION To simplify the arithmetic, we want a 1 in the upper-left corner. We could interchange rows 1 and 4. Instead, we factor out 2 from the top row, and then proceed with row replacements in the first column:

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

Next, we could factor out another 2 from row 3 or use the 3 in the second column as a pivot. We choose the latter operation, adding 4 times row 2 to row 3:

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

Finally, adding $-1/2$ times row 3 to row 4, and computing the “triangular” determinant, we find that

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \cdot (1)(3)(-6)(1) = -36 \quad \blacksquare$$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \quad \det U \neq 0$$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \det U = 0$$

FIGURE 1
Typical echelon forms of square matrices.

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. (This is always possible. See the row reduction algorithm in Section 1.2.) If there are r interchanges, then Theorem 3 shows that

$$\det A = (-1)^r \det U$$

Since U is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries u_{11}, \dots, u_{nn} . If A is invertible, the entries u_{ii} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's). Otherwise, at least u_{nn} is zero, and the product $u_{11} \cdots u_{nn}$ is zero. See Fig. 1. Thus

$$\det A = \begin{cases} (-1)^r \cdot \left(\text{product of} \right. \\ \left. \text{pivots in } U \right) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases} \quad (1)$$

It is interesting to note that although the echelon form U described above is not unique (because it is not completely row reduced), and the pivots are not unique, the *product* of the pivots *is* unique, except for a possible minus sign.

Formula (1) not only gives a concrete interpretation of $\det A$ but also proves the main theorem of this section:

THEOREM 4

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4 adds the statement “ $\det A \neq 0$ ” to the Invertible Matrix Theorem. A useful corollary is that $\det A = 0$ when the columns of A are linearly dependent. Also, $\det A = 0$ when the rows of A are linearly dependent. (Rows of A are columns of A^T , and linearly dependent columns of A^T make A^T singular. When A^T is singular, so is A , by the Invertible Matrix Theorem.) In practice, linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

EXAMPLE 3 Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

SOLUTION Add 2 times row 1 to row 3 to obtain

$$\det A = \det \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

because the second and third rows of the second matrix are equal. ■

NUMERICAL NOTES

1. Most computer programs that compute $\det A$ for a general matrix A use the method of formula (1) above.
2. It can be shown that evaluation of an $n \times n$ determinant using row operations requires about $2n^3/3$ arithmetic operations. Any modern microcomputer can calculate a 25×25 determinant in a fraction of a second, since only about 10,000 operations are required.

WEB

Computers can also handle large “sparse” matrices, with special routines that take advantage of the presence of many zeros. Of course, zero entries can speed hand computations, too. The calculations in the next example combine the power of row operations with the strategy from Section 3.1 of using zero entries in cofactor expansions.

EXAMPLE 4 Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

SOLUTION A good way to begin is to use the 2 in column 1 as a pivot, eliminating the -2 below it. Then use a cofactor expansion to reduce the size of the determinant, followed by another row replacement operation. Thus

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

An interchange of rows 2 and 3 would produce a “triangular determinant.” Another approach is to make a cofactor expansion down the first column:

$$\det A = (-2)(1) \begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2 \cdot (15) = -30 \quad \blacksquare$$

Column Operations

We can perform operations on the columns of a matrix in a way that is analogous to the row operations we have considered. The next theorem shows that column operations have the same effects on determinants as row operations.

THEOREM 5

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

PROOF The theorem is obvious for $n = 1$. Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$. Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T , because the cofactors involve $k \times k$ determinants. Hence the cofactor expansion of $\det A$ along the first row equals the cofactor expansion of $\det A^T$ down the first column. That is, A and A^T have equal determinants. Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of n implies its truth for the next value of n . By the principle of induction, the theorem is true for all $n \geq 1$. \blacksquare

Because of Theorem 5, each statement in Theorem 3 is true when the word *row* is replaced everywhere by *column*. To verify this property, one merely applies the original Theorem 3 to A^T . A row operation on A^T amounts to a column operation on A .

Column operations are useful for both theoretical purposes and hand computations. However, for simplicity we’ll perform only row operations in numerical calculations.

Determinants and Matrix Products

The proof of the following useful theorem is at the end of the section. Applications are in the exercises.

THEOREM 6

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

EXAMPLE 5 Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

SOLUTION

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since $\det A = 9$ and $\det B = 5$,

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB \quad \blacksquare$$

Warning: A common misconception is that Theorem 6 has an analogue for *sums* of matrices. However, $\det(A + B)$ is *not* equal to $\det A + \det B$, in general.

A Linearity Property of the Determinant Function

For an $n \times n$ matrix A , we can consider $\det A$ as a function of the n column vectors in A . We will show that if all columns except one are held fixed, then $\det A$ is a *linear function* of that one (vector) variable.

Suppose that the j th column of A is allowed to vary, and write

$$A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{j-1} \quad \mathbf{x} \quad \mathbf{a}_{j+1} \quad \cdots \quad \mathbf{a}_n]$$

Define a transformation T from \mathbb{R}^n to \mathbb{R} by

$$T(\mathbf{x}) = \det[\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{j-1} \quad \mathbf{x} \quad \mathbf{a}_{j+1} \quad \cdots \quad \mathbf{a}_n]$$

Then,

$$T(c\mathbf{x}) = cT(\mathbf{x}) \quad \text{for all scalars } c \text{ and all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (2)$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbb{R}^n \quad (3)$$

Property (2) is Theorem 3(c) applied to the columns of A . A proof of property (3) follows from a cofactor expansion of $\det A$ down the j th column. (See Exercise 43.) This (multi-) linearity property of the determinant turns out to have many useful consequences that are studied in more advanced courses.

Proofs of Theorems 3 and 6

It is convenient to prove Theorem 3 when it is stated in terms of the elementary matrices discussed in Section 2.2. We call an elementary matrix E a *row replacement (matrix)* if E is obtained from the identity I by adding a multiple of one row to another row; E is an *interchange* if E is obtained by interchanging two rows of I ; and E is a *scale by r* if E is obtained by multiplying a row of I by a nonzero scalar r . With this terminology, Theorem 3 can be reformulated as follows:

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

PROOF OF THEOREM 3 The proof is by induction on the size of A . The case of a 2×2 matrix was verified in Exercises 33–36 of Section 3.1. Suppose the theorem has been verified for determinants of $k \times k$ matrices with $k \geq 2$, let $n = k + 1$, and let A be $n \times n$. The action of E on A involves either two rows or only one row. So we can expand $\det EA$ across a row that is unchanged by the action of E , say, row i . Let A_{ij} (respectively, B_{ij}) be the matrix obtained by deleting row i and column j from A (respectively, EA). Then the rows of B_{ij} are obtained from the rows of A_{ij} by the same type of elementary row operation that E performs on A . Since these submatrices are only $k \times k$, the induction assumption implies that

$$\det B_{ij} = \alpha \cdot \det A_{ij}$$

where $\alpha = 1, -1$, or r , depending on the nature of E . The cofactor expansion across row i is

$$\begin{aligned} \det EA &= a_{i1}(-1)^{i+1} \det B_{i1} + \cdots + a_{in}(-1)^{i+n} \det B_{in} \\ &= \alpha a_{i1}(-1)^{i+1} \det A_{i1} + \cdots + \alpha a_{in}(-1)^{i+n} \det A_{in} \\ &= \alpha \cdot \det A \end{aligned}$$

In particular, taking $A = I_n$, we see that $\det E = 1, -1$, or r , depending on the nature of E . Thus the theorem is true for $n = 2$, and the truth of the theorem for one value of n implies its truth for the next value of n . By the principle of induction, the theorem must be true for $n \geq 2$. The theorem is trivially true for $n = 1$. ■

PROOF OF THEOREM 6 If A is not invertible, then neither is AB , by Exercise 27 in Section 2.3. In this case, $\det AB = (\det A)(\det B)$, because both sides are zero, by Theorem 4. If A is invertible, then A and the identity matrix I_n are row equivalent by the Invertible Matrix Theorem. So there exist elementary matrices E_1, \dots, E_p such that

$$A = E_p E_{p-1} \cdots E_1 \cdot I_n = E_p E_{p-1} \cdots E_1$$

For brevity, write $|A|$ for $\det A$. Then repeated application of Theorem 3, as rephrased above, shows that

$$\begin{aligned} |AB| &= |E_p \cdots E_1 B| = |E_p| |E_{p-1} \cdots E_1 B| = \cdots \\ &= |E_p| \cdots |E_1| |B| = \cdots = |E_p \cdots E_1| |B| \\ &= |A| |B| \end{aligned}$$

PRACTICE PROBLEMS

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.

2. Use a determinant to decide if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

3.2 EXERCISES

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

$$1. \begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$$

$$2. \begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$$

$$3. \begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$$

$$4. \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 1 & -5 \end{vmatrix}$$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

$$5. \begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

$$6. \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$$

$$7. \begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$$

$$8. \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}$$

$$9. \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

$$10. \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

$$11. \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix}$$

$$12. \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$$

$$13. \begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

$$14. \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$$

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

$$15. \begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$$

$$16. \begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$$

$$17. \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

$$18. \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$19. \begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$$

$$20. \begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

$$21. \begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

$$22. \begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

$$23. \begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

$$24. \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix} \quad 25. \begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

$$26. \begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}$$

In Exercises 27 and 28, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

27. a. A row replacement operation does not affect the determinant of a matrix.
 b. The determinant of A is the product of the pivots in any echelon form U of A , multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U .

- c. If the columns of A are linearly dependent, then $\det A = 0$.
- d. $\det(A + B) = \det A + \det B$.
28. a. If two row interchanges are made in succession, then the new determinant equals the old determinant.
- b. The determinant of A is the product of the diagonal entries in A .
- c. If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.
- d. $\det A^T = (-1) \det A$.
29. Compute $\det B^5$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.
30. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix A are equal, then $\det A = 0$. The same is true for two columns. Why?

In Exercises 31–36, mention an appropriate theorem in your explanation.

31. Show that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.
32. Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.
33. Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.
34. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.
35. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.
36. Suppose that A is a square matrix such that $\det A^4 = 0$. Explain why A cannot be invertible.

Verify that $\det AB = (\det A)(\det B)$ for the matrices in Exercises 37 and 38. (Do not use Theorem 6.)

37. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}$

39. Let A and B be 3×3 matrices, with $\det A = 4$ and $\det B = -3$. Use properties of determinants (in the text and

in the exercises above) to compute:

- a. $\det AB$ b. $\det 5A$ c. $\det B^T$
 d. $\det A^{-1}$ e. $\det A^3$

40. Let A and B be 4×4 matrices, with $\det A = -1$ and $\det B = 2$. Compute:
 a. $\det AB$ b. $\det B^5$ c. $\det 2A$
 d. $\det A^T A$ e. $\det B^{-1} AB$

41. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$$

42. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

43. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix},$$

$$B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

Note, however, that A is *not* the same as $B + C$.

44. Right-multiplication by an elementary matrix E affects the *columns* of A in the same way that left-multiplication affects the *rows*. Use Theorems 5 and 3 and the obvious fact that E^T is another elementary matrix to show that

$$\det AE = (\det E)(\det A)$$

Do not use Theorem 6.

45. [M] Compute $\det A^T A$ and $\det AA^T$ for several random 4×5 matrices and several random 5×6 matrices. What can you say about $A^T A$ and AA^T when A has more columns than rows?

46. [M] If $\det A$ is close to zero, is the matrix A nearly singular? Experiment with the nearly singular 4×4 matrix A in Exercise 9 of Section 2.3. Compute the determinants of A , $10A$, and $0.1A$. In contrast, compute the condition numbers of these matrices. Repeat these calculations when A is the 4×4 identity matrix. Discuss your results.

SOLUTIONS TO PRACTICE PROBLEMS

1. Perform row replacements to create zeros in the first column and then create a row of zeros.

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{aligned}
 2. \det[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] &= \begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix} = \begin{vmatrix} 5 & -3 & 2 \\ -2 & 0 & -5 \\ 9 & -5 & 5 \end{vmatrix} && \text{Row 1 added} \\
 &&& \text{to row 2} \\
 &= -(-3) \begin{vmatrix} -2 & -5 \\ 9 & 5 \end{vmatrix} - (-5) \begin{vmatrix} 5 & 2 \\ -2 & -5 \end{vmatrix} && \text{Cofactors of} \\
 &= 3 \cdot (35) + 5 \cdot (-21) = 0 && \text{column 2}
 \end{aligned}$$

By Theorem 4, the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is not invertible. The columns are linearly dependent, by the Invertible Matrix Theorem.

3.3 CRAMER'S RULE, VOLUME, AND LINEAR TRANSFORMATIONS

This section applies the theory of the preceding sections to obtain important theoretical formulas and a geometric interpretation of the determinant.

Cramer's Rule

Cramer's rule is needed in a variety of theoretical calculations. For instance, it can be used to study how the solution of $A\mathbf{x} = \mathbf{b}$ is affected by changes in the entries of \mathbf{b} . However, the formula is inefficient for hand calculations, except for 2×2 or perhaps 3×3 matrices.

For any $n \times n$ matrix A and any \mathbf{b} in \mathbb{R}^n , let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector \mathbf{b} .

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \mathbf{b} \quad \cdots \quad \mathbf{a}_n]$$

\uparrow
 col i

THEOREM 7

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any \mathbf{b} in \mathbb{R}^n , the unique solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots, n \quad (1)$$

PROOF Denote the columns of A by $\mathbf{a}_1, \dots, \mathbf{a}_n$ and the columns of the $n \times n$ identity matrix I by $\mathbf{e}_1, \dots, \mathbf{e}_n$. If $A\mathbf{x} = \mathbf{b}$, the definition of matrix multiplication shows that

$$\begin{aligned}
 A \cdot I_i(\mathbf{x}) &= A [\mathbf{e}_1 \quad \cdots \quad \mathbf{x} \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad \cdots \quad A\mathbf{x} \quad \cdots \quad A\mathbf{e}_n] \\
 &= [\mathbf{a}_1 \quad \cdots \quad \mathbf{b} \quad \cdots \quad \mathbf{a}_n] = A_i(\mathbf{b})
 \end{aligned}$$

By the multiplicative property of determinants,

$$(\det A)(\det I_i(\mathbf{x})) = \det A_i(\mathbf{b})$$

The second determinant on the left is simply x_i . (Make a cofactor expansion along the i th row.) Hence $(\det A) \cdot x_i = \det A_i(\mathbf{b})$. This proves (1) because A is invertible and $\det A \neq 0$. ■

EXAMPLE 1 Use Cramer's rule to solve the system

$$\begin{aligned}
 3x_1 - 2x_2 &= 6 \\
 -5x_1 + 4x_2 &= 8
 \end{aligned}$$

SOLUTION View the system as $A\mathbf{x} = \mathbf{b}$. Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

Since $\det A = 2$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{24 + 30}{2} = 27 \quad \blacksquare$$

Application to Engineering

A number of important engineering problems, particularly in electrical engineering and control theory, can be analyzed by *Laplace transforms*. This approach converts an appropriate system of linear differential equations into a system of linear algebraic equations whose coefficients involve a parameter. The next example illustrates the type of algebraic system that may arise.

EXAMPLE 2 Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$3sx_1 - 2x_2 = 4$$

$$-6x_1 + sx_2 = 1$$

SOLUTION View the system as $A\mathbf{x} = \mathbf{b}$. Then

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

Since

$$\det A = 3s^2 - 12 = 3(s + 2)(s - 2)$$

the system has a unique solution precisely when $s \neq \pm 2$. For such an s , the solution is (x_1, x_2) , where

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{4s + 2}{3(s + 2)(s - 2)}$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{3s + 24}{3(s + 2)(s - 2)} = \frac{s + 8}{(s + 2)(s - 2)} \quad \blacksquare$$

A Formula for A^{-1}

Cramer's rule leads easily to a general formula for the inverse of an $n \times n$ matrix A . The j th column of A^{-1} is a vector \mathbf{x} that satisfies

$$A\mathbf{x} = \mathbf{e}_j$$

where \mathbf{e}_j is the j th column of the identity matrix, and the i th entry of \mathbf{x} is the (i, j) -entry of A^{-1} . By Cramer's rule,

$$\{(i, j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A} \quad (2)$$

Recall that A_{ji} denotes the submatrix of A formed by deleting row j and column i . A cofactor expansion down column i of $A_i(\mathbf{e}_j)$ shows that

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji} \quad (3)$$

where C_{ji} is a cofactor of A . By (2), the (i, j) -entry of A^{-1} is the cofactor C_{ji} divided by $\det A$. [Note that the subscripts on C_{ji} are the reverse of (i, j) .] Thus

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \quad (4)$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or **classical adjoint**) of A , denoted by $\text{adj } A$. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).

THEOREM 8

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

EXAMPLE 3 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

SOLUTION The nine cofactors are

$$\begin{aligned} C_{11} &= + \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, & C_{12} &= - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, & C_{13} &= + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5 \\ C_{21} &= - \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, & C_{22} &= + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, & C_{23} &= - \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7 \\ C_{31} &= + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, & C_{32} &= - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, & C_{33} &= + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \end{aligned}$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, C_{12} goes in the $(2, 1)$ position.] Thus

$$\text{adj } A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute $\det A$ directly, but the following computation provides a check on the calculations above *and* produces $\det A$:

$$(\text{adj } A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since $(\text{adj } A)A = 14I$, Theorem 8 shows that $\det A = 14$ and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix} \quad \blacksquare$$

NUMERICAL NOTES

Theorem 8 is useful mainly for theoretical calculations. The formula for A^{-1} permits one to deduce properties of the inverse without actually calculating it. Except for special cases, the algorithm in Section 2.2 gives a much better way to compute A^{-1} , if the inverse is really needed.

Cramer's rule is also a theoretical tool. It can be used to study how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ is to changes in an entry in \mathbf{b} or in A (perhaps due to experimental error when acquiring the entries for \mathbf{b} or A). When A is a 3×3 matrix with *complex* entries, Cramer's rule is sometimes selected for hand computation because row reduction of $[A \ \mathbf{b}]$ with complex arithmetic can be messy, and the determinants are fairly easy to compute. For a larger $n \times n$ matrix (real or complex), Cramer's rule is hopelessly inefficient. Computing just *one* determinant takes about as much work as solving $A\mathbf{x} = \mathbf{b}$ by row reduction.

Determinants as Area or Volume

In the next application, we verify the geometric interpretation of determinants described in the chapter introduction. Although a general discussion of length and distance in \mathbb{R}^n will not be given until Chapter 6, we assume here that the usual Euclidean concepts of length, area, and volume are already understood for \mathbb{R}^2 and \mathbb{R}^3 .

THEOREM 9

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

SG

A Geometric Proof
3-12

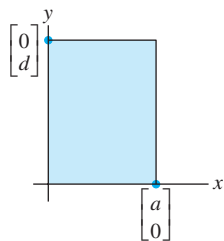


FIGURE 1
Area = $|ad|$.

PROOF The theorem is obviously true for any 2×2 diagonal matrix:

$$\left| \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad| = \begin{cases} \text{area of} \\ \text{rectangle} \end{cases}$$

See Fig. 1. It will suffice to show that any 2×2 matrix $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$. From Section 3.2, we know that the absolute value of the determinant is unchanged when two columns are interchanged or a multiple of one column is added to another. And it is easy to see that such operations suffice to transform A into a diagonal matrix. Column interchanges do not change the parallelogram at all. So it suffices to prove the following simple geometric observation that applies to vectors in \mathbb{R}^2 or \mathbb{R}^3 :

Let \mathbf{a}_1 and \mathbf{a}_2 be nonzero vectors. Then for any scalar c , the area of the parallelogram determined by \mathbf{a}_1 and \mathbf{a}_2 equals the area of the parallelogram determined by \mathbf{a}_1 and $\mathbf{a}_2 + c\mathbf{a}_1$.

To prove this statement, we may assume that \mathbf{a}_2 is not a multiple of \mathbf{a}_1 , for otherwise the two parallelograms would be degenerate and have zero area. If L is the line through $\mathbf{0}$ and \mathbf{a}_1 , then $\mathbf{a}_2 + L$ is the line through \mathbf{a}_2 parallel to L , and $\mathbf{a}_2 + c\mathbf{a}_1$ is on this line. See Fig. 2. The points \mathbf{a}_2 and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to L . Hence the two parallelograms in Fig. 2 have the same area, since they share the base from $\mathbf{0}$ to \mathbf{a}_1 . This completes the proof for \mathbb{R}^2 .

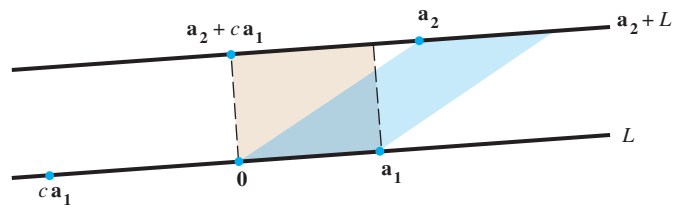


FIGURE 2 Two parallelograms of equal area.

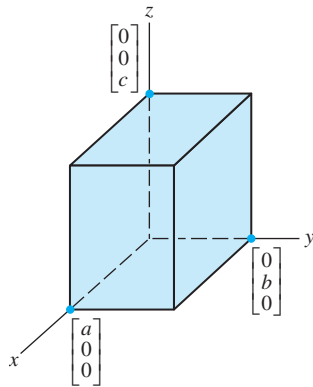


FIGURE 3
Volume = $|abc|$.

The proof for \mathbb{R}^3 is similar. The theorem is obviously true for a 3×3 diagonal matrix. See Fig. 3. And any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det A|$. (Think about doing row operations on A^T .) So it suffices to show that these operations do not affect the volume of the parallelepiped determined by the columns of A .

A parallelepiped is shown in Fig. 4 as a shaded box with two sloping sides. Its volume is the area of the base in the plane $\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$ times the altitude of \mathbf{a}_2 above $\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$. Any vector $\mathbf{a}_2 + c\mathbf{a}_1$ has the same altitude because $\mathbf{a}_2 + c\mathbf{a}_1$ lies in the plane $\mathbf{a}_2 + \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$, which is parallel to $\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$. Hence the volume of the parallelepiped is unchanged when $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ is changed to $[\mathbf{a}_1 \ \mathbf{a}_2 + c\mathbf{a}_1 \ \mathbf{a}_3]$. Thus a column replacement operation does not affect the volume of the parallelepiped. Since column interchanges have no effect on the volume, the proof is complete. ■

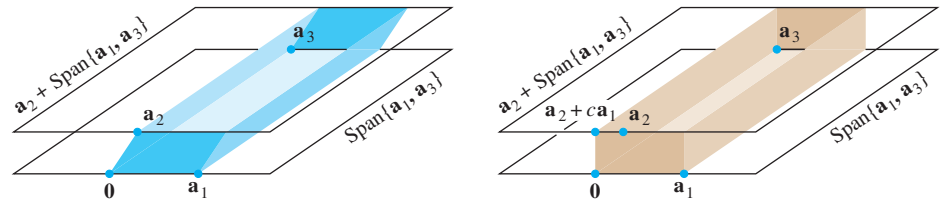


FIGURE 4 Two parallelepipeds of equal volume.

EXAMPLE 4 Calculate the area of the parallelogram determined by the points $(-2, -2)$, $(0, 3)$, $(4, -1)$, and $(6, 4)$. See Fig. 5(a).

SOLUTION First translate the parallelogram to one having the origin as a vertex. For example, subtract the vertex $(-2, -2)$ from each of the four vertices. The new parallelogram has the same area, and its vertices are $(0, 0)$, $(2, 5)$, $(6, 1)$, and $(8, 6)$. See

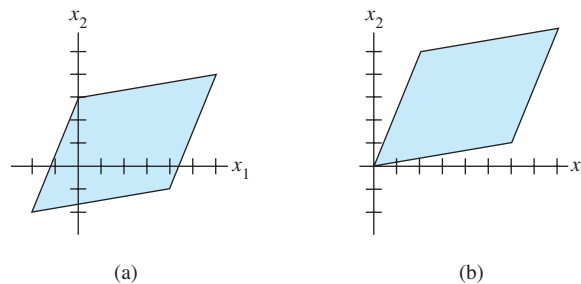


FIGURE 5 Translating a parallelogram does not change its area.

Fig. 5(b). This parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

Since $|\det A| = |-28|$, the area of the parallelogram is 28. ■

Linear Transformations

Determinants can be used to describe an important geometric property of linear transformations in the plane and in \mathbb{R}^3 . If T is a linear transformation and S is a set in the domain of T , let $T(S)$ denote the set of images of points in S . We are interested in how the area (or volume) of $T(S)$ compares with the area (or volume) of the original set S . For convenience, when S is a region bounded by a parallelogram, we also refer to S as a parallelogram.

THEOREM 10

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A . If S is a parallelogram in \mathbb{R}^2 , then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \quad (5)$$

If T is determined by a 3×3 matrix A , and if S is a parallelepiped in \mathbb{R}^3 , then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \quad (6)$$

PROOF Consider the 2×2 case, with $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$. A parallelogram at the origin in \mathbb{R}^2 determined by vectors \mathbf{b}_1 and \mathbf{b}_2 has the form

$$S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\}$$

The image of S under T consists of points of the form

$$\begin{aligned} T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) &= s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2) \\ &= s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2 \end{aligned}$$

where $0 \leq s_1 \leq 1$, $0 \leq s_2 \leq 1$. It follows that $T(S)$ is the parallelogram determined by the columns of the matrix $[A\mathbf{b}_1 \quad A\mathbf{b}_2]$. This matrix can be written as AB , where $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$. By Theorem 9 and the product theorem for determinants,

$$\begin{aligned} \{\text{area of } T(S)\} &= |\det AB| = |\det A| \cdot |\det B| \\ &= |\det A| \cdot \{\text{area of } S\} \end{aligned} \quad (7)$$

An arbitrary parallelogram has the form $\mathbf{p} + S$, where \mathbf{p} is a vector and S is a parallelogram at the origin, as above. It is easy to see that T transforms $\mathbf{p} + S$ into $T(\mathbf{p}) + T(S)$. (See Exercise 26.) Since translation does not affect the area of a set,

$$\begin{aligned} \{\text{area of } T(\mathbf{p} + S)\} &= \{\text{area of } T(\mathbf{p}) + T(S)\} \\ &= \{\text{area of } T(S)\} && \text{Translation} \\ &= |\det A| \cdot \{\text{area of } S\} && \text{By equation (7)} \\ &= |\det A| \cdot \{\text{area of } \mathbf{p} + S\} && \text{Translation} \end{aligned}$$

This shows that (5) holds for all parallelograms in \mathbb{R}^2 . The proof of (6) for the 3×3 case is analogous. ■

When we attempt to generalize Theorem 10 to a region in \mathbb{R}^2 or \mathbb{R}^3 that is not bounded by straight lines or planes, we must face the problem of how to define and compute its area or volume. This is a question studied in calculus, and we shall only outline the basic idea for \mathbb{R}^2 . If R is a planar region that has a finite area, then R can be approximated by a grid of small squares that lie inside R . By making the squares sufficiently small, the area of R may be approximated as closely as desired by the sum of the areas of the small squares. See Fig. 6.

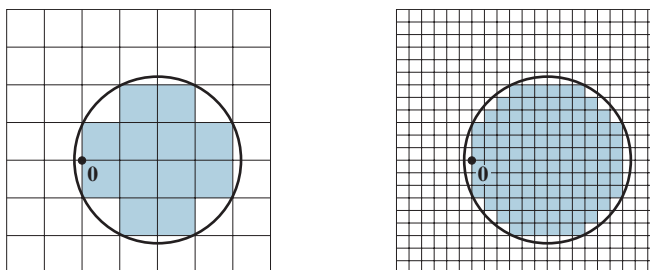


FIGURE 6 Approximating a planar region by a union of squares. The approximation improves as the grid becomes finer.

If T is a linear transformation associated with a 2×2 matrix A , then the image of a planar region R under T is approximated by the images of the small squares inside R . The proof of Theorem 10 shows that each such image is a parallelogram whose area is $|\det A|$ times the area of the square. If R' is the union of the squares inside R , then the area of $T(R')$ is $|\det A|$ times the area of R' . See Fig. 7. Also, the area of $T(R')$ is close to the area of $T(R)$. An argument involving a limiting process may be given to justify the following generalization of Theorem 10.

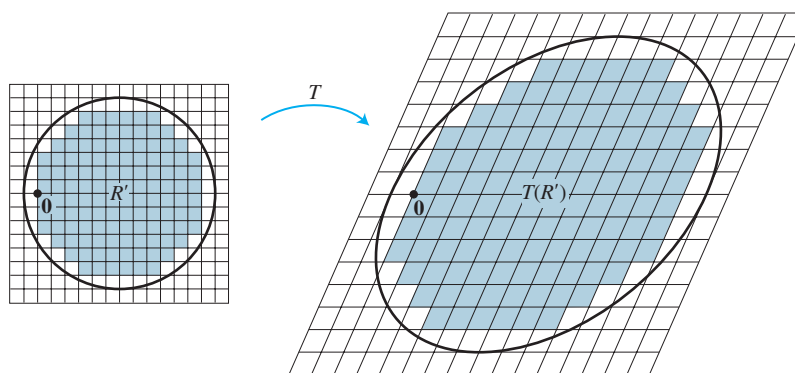
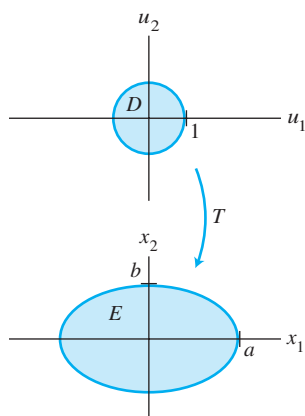


FIGURE 7 Approximating $T(R)$ by a union of parallelograms.

The conclusions of Theorem 10 hold whenever S is a region in \mathbb{R}^2 with finite area or a region in \mathbb{R}^3 with finite volume.

EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$



SOLUTION We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, because if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{x} = A\mathbf{u}$, then

$$u_1 = \frac{x_1}{a} \quad \text{and} \quad u_2 = \frac{x_2}{b}$$

It follows that \mathbf{u} is in the unit disk, with $u_1^2 + u_2^2 \leq 1$, if and only if \mathbf{x} is in E , with $(x_1/a)^2 + (x_2/b)^2 \leq 1$. By the generalization of Theorem 10,

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi(1)^2 = \pi ab \end{aligned}$$

PRACTICE PROBLEM

Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

3.3 EXERCISES

Use Cramer's rule to compute the solutions of the systems in Exercises 1–6.

1. $5x_1 + 7x_2 = 3$
 $2x_1 + 4x_2 = 1$
2. $4x_1 + x_2 = 6$
 $5x_1 + 2x_2 = 7$
3. $3x_1 - 2x_2 = 7$
 $-5x_1 + 6x_2 = -5$
4. $-5x_1 + 3x_2 = 9$
 $3x_1 - x_2 = -5$
5. $2x_1 + x_2 = 7$
 $-3x_1 + x_3 = -8$
 $x_2 + 2x_3 = -3$
6. $2x_1 + x_2 + x_3 = 4$
 $-x_1 + 2x_3 = 2$
 $3x_1 + x_2 + 3x_3 = -2$

In Exercises 7–10, determine the values of the parameter s for which the system has a unique solution, and describe the solution.

7. $6sx_1 + 4x_2 = 5$
 $9x_1 + 2sx_2 = -2$
8. $3sx_1 - 5x_2 = 3$
 $9x_1 + 5sx_2 = 2$
9. $sx_1 - 2sx_2 = -1$
 $3x_1 + 6sx_2 = 4$
10. $2sx_1 + x_2 = 1$
 $3sx_1 + 6sx_2 = 2$

In Exercises 11–16, compute the adjugate of the given matrix, and then use Theorem 8 to give the inverse of the matrix.

11. $\begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$
12. $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
13. $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$
14. $\begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$$15. \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{bmatrix} \qquad 16. \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

17. Show that if A is 2×2 , then Theorem 8 gives the same formula for A^{-1} as that given by Theorem 4 in Section 2.2.

18. Suppose that all the entries in A are integers and $\det A = 1$. Explain why all the entries in A^{-1} are integers.

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0, 0)$, $(5, 2)$, $(6, 4)$, $(11, 6)$
20. $(0, 0)$, $(-1, 3)$, $(4, -5)$, $(3, -2)$
21. $(-1, 0)$, $(0, 5)$, $(1, -4)$, $(2, 1)$
22. $(0, -2)$, $(6, -1)$, $(-3, 1)$, $(3, 2)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -2)$, $(1, 2, 4)$, and $(7, 1, 0)$.

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 4, 0)$, $(-2, -5, 2)$, and $(-1, 2, -1)$.

25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [Hint: Think about the columns of A .]

26. Let $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation, and let \mathbf{p} be a vector and S a set in \mathbb{R}^m . Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in \mathbb{R}^n .

27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}$.

29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, \mathbf{v}_1 , and \mathbf{v}_2 in \mathbb{R}^2 .

30. Let R be the triangle with vertices at (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

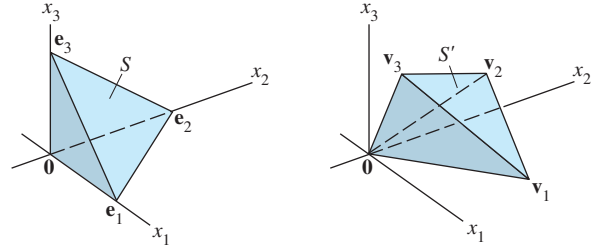
[Hint: Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a , b , and c are

positive numbers. Let S be the unit ball, whose bounding surface has the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

- Show that $T(S)$ is bounded by the ellipsoid with the equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$.
- Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by the ellipsoid in part (a).

32. Let S be the tetrahedron in \mathbb{R}^3 with vertices at the vectors $\mathbf{0}$, \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 , and let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . See the figure.



- Describe a linear transformation that maps S onto S' .
- Find a formula for the volume of the tetrahedron S' using the fact that

$$\{\text{volume of } S\} = (1/3)\{\text{area of base}\} \cdot \{\text{height}\}$$

33. [M] Test the inverse formula of Theorem 8 for a random 4×4 matrix A . Use your matrix program to compute the cofactors of the 3×3 submatrices, construct the adjugate, and set $B = (\text{adj } A)/(\det A)$. Then compute $B - \text{inv}(A)$, where $\text{inv}(A)$ is the inverse of A as computed by the matrix program. Use floating point arithmetic with the maximum possible number of decimal places. Report your results.

34. [M] Test Cramer's rule for a random 4×4 matrix A and a random 4×1 vector \mathbf{b} . Compute each entry in the solution of $A\mathbf{x} = \mathbf{b}$, and compare these entries with the entries in $A^{-1}\mathbf{b}$. Write the command (or keystrokes) for your matrix program that uses Cramer's rule to produce the second entry of \mathbf{x} .

35. [M] If your version of MATLAB has the `flops` command, use it to count the number of floating point operations to compute A^{-1} for a random 30×30 matrix. Compare this number with the number of flops needed to form $(\text{adj } A)/(\det A)$.

SOLUTION TO PRACTICE PROBLEM

The area of S is $\left| \det \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} \right| = 14$, and $\det A = 2$. By Theorem 10, the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is

$$|\det A| \cdot \{\text{area of } S\} = 2 \cdot 14 = 28$$

CHAPTER 3 SUPPLEMENTARY EXERCISES

- Mark each statement True or False. Justify each answer. Assume that all matrices here are square.
 - If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.
 - If two rows of a 3×3 matrix A are the same, then $\det A = 0$.
 - If A is a 3×3 matrix, then $\det 5A = 5 \det A$.
 - If A and B are $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A + B) = 5$.
 - If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$.
 - If B is produced by interchanging two rows of A , then $\det B = \det A$.
 - If B is produced by multiplying row 3 of A by 5, then $\det B = 5 \cdot \det A$.

- h. If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.
- i. $\det A^T = \det A$.
- j. $\det(-A) = -\det A$.
- k. $\det A^T A \geq 0$.
- l. Any system of n linear equations in n variables can be solved by Cramer's rule.
- m. If \mathbf{u} and \mathbf{v} are in \mathbb{R}^2 and $\det[\mathbf{u} \ \mathbf{v}] = 10$, then the area of the triangle in the plane with vertices at $\mathbf{0}$, \mathbf{u} , and \mathbf{v} is 10.
- n. If $A^3 = 0$, then $\det A = 0$.
- o. If A is invertible, then $\det A^{-1} = \det A$.
- p. If A is invertible, then $(\det A)(\det A^{-1}) = 1$.

Use row operations to show that the determinants in Exercises 2–4 are all zero.

$$2. \begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix} \qquad 3. \begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$$

$$4. \begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix}$$

Compute the determinants in Exercises 5 and 6.

$$5. \begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix}$$

$$6. \begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix}$$

7. Show that the equation of the line in \mathbb{R}^2 through distinct points (x_1, y_1) and (x_2, y_2) can be written as

$$\det \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = 0$$

8. Find a 3×3 determinant equation similar to that in Exercise 7 that describes the equation of the line through (x_1, y_1) with slope m .

Exercises 9 and 10 concern determinants of the following *Vandermonde matrices*.

$$T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}, \quad V(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

- 9. Use row operations to show that $\det T = (b-a)(c-a)(c-b)$.
- 10. Let $f(t) = \det V$, with x_1, x_2, x_3 all distinct. Explain why $f(t)$ is a cubic polynomial, show that the coefficient of t^3 is nonzero, and find three points on the graph of f .
- 11. Determine the area of the parallelogram determined by the points $(1, 4)$, $(-1, 5)$, $(3, 9)$, and $(5, 8)$. How can you tell that the quadrilateral determined by the points is actually a parallelogram?
- 12. Use the concept of area of a parallelogram to write a statement about a 2×2 matrix A that is true if and only if A is invertible.
- 13. Show that if A is invertible, then $\text{adj } A$ is invertible, and

$$(\text{adj } A)^{-1} = \frac{1}{\det A} A$$

[Hint: Given matrices B and C , what calculation(s) would show that C is the inverse of B ?]

14. Let A, B, C, D , and I be $n \times n$ matrices. Use the definition or properties of a determinant to justify the following formulas. Part (c) is useful in applications of eigenvalues (Chapter 5).

$$a. \det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \det A \qquad b. \det \begin{bmatrix} I & 0 \\ C & D \end{bmatrix} = \det D$$

$$c. \det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = (\det A)(\det D) = \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

15. Let A, B, C , and D be $n \times n$ matrices with A invertible.
- a. Find matrices X and Y to produce the block LU factorization $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$ and then show that $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A) \cdot \det(D - CA^{-1}B)$
 - b. Show that if $AC = CA$, then $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB)$

16. Let J be the $n \times n$ matrix of all 1's, and consider $A = (a-b)I + bJ$; that is,

$$A = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

Confirm that $\det A = (a-b)^{n-1}[a + (n-1)b]$ as follows:

- a. Subtract row 2 from row 1, row 3 from row 2, and so on, and explain why this does not change the determinant of the matrix.

- b. With the resulting matrix from part (a), add column 1 to column 2, then add this new column 2 to column 3, and so on, and explain why this does not change the determinant.
 c. Find the determinant of the resulting matrix from (b).

17. Let A be the original matrix given in Exercise 16, and let

$$B = \begin{bmatrix} a-b & b & b & \cdots & b \\ 0 & a & b & \cdots & b \\ 0 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b & b & \cdots & a \end{bmatrix},$$

$$C = \begin{bmatrix} b & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

Notice that A , B , and C are nearly the same except that the first column of A equals the sum of the first columns of B and C . A *linearity property* of the determinant function, discussed in Section 3.2, says that $\det A = \det B + \det C$. Use this fact to prove the formula in Exercise 16 by induction on the size of matrix A .

18. [M] Apply the result of Exercise 16 to find the determinants of the following matrices, and confirm your answers using a matrix program.

$$\begin{bmatrix} 3 & 8 & 8 & 8 \\ 8 & 3 & 8 & 8 \\ 8 & 8 & 3 & 8 \\ 8 & 8 & 8 & 3 \end{bmatrix} \quad \begin{bmatrix} 8 & 3 & 3 & 3 & 3 \\ 3 & 8 & 3 & 3 & 3 \\ 3 & 3 & 8 & 3 & 3 \\ 3 & 3 & 3 & 8 & 3 \\ 3 & 3 & 3 & 3 & 8 \end{bmatrix}$$

19. [M] Use a matrix program to compute the determinants of the following matrices.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Use the results to guess the determinant of the matrix below, and confirm your guess by using row operations to evaluate that determinant.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

20. [M] Use the method of Exercise 19 to guess the determinant of

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & \cdots & 3 \\ 1 & 3 & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 6 & \cdots & 3(n-1) \end{bmatrix}$$

Justify your conjecture. [Hint: Use Exercise 14(c) and the result of Exercise 19.]

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