- $\frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty x^{\alpha-1}e^{-\lambda x}\,dx$

Beta

- $\frac{1}{\mathrm{B}(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$
- $\mathbb{E}\left[X^k\right] = \frac{\mathrm{B}(\alpha+k,\beta)}{\mathrm{B}(\alpha,\beta)} = \frac{\alpha+k-1}{\alpha+\beta+k-1}\mathbb{E}\left[X^{k-1}\right]$
- $\mathrm{Beta}\,(1,1) \sim \mathrm{Unif}\,(0,1)$

8 Probability and Moment Generating Functions

- $G_X(t) = \mathbb{E}\left[t^X\right] \qquad |t| < 1$
- $M_X(t) = G_X(e^t) = \mathbb{E}\left[e^{Xt}\right] = \mathbb{E}\left[\sum_{i=0}^\infty \frac{(Xt)^i}{i!}\right] = \sum_{i=0}^\infty \frac{\mathbb{E}\left[X^i\right]}{i!}\cdot t^i$
- $\mathbb{P}\left[X=0\right] = G_X(0)$
- $\mathbb{P}\left[X=1\right] = G'_X(0)$
- $\mathbb{P}\left[X=i\right] = \frac{G_X^{(i)}(0)}{i!}$
- $\mathbb{E}\left[X\right] = G'_X(1^-)$
- $\mathbb{E}\left[X^k\right] = M_X^{(k)}(0)$
- $\mathbb{E}\left[\frac{X!}{(X-k)!}\right] = G_X^{(k)}(1^-)$
- $\mathbb{V}\left[X\right] = G''_X(1^-) + G'_X(1^-) - \left(G'_X(1^-)\right)^2$
- $G_X(t) = G_Y(t) \implies X \overset{d}{=} Y$

9 Multivariate Distributions

9.1 Standard Bivariate Normal

Let $X, Y \sim \mathcal{N}(0,1) \wedge X \perp\!\!\!\perp Z$ where $Y = \rho X + \sqrt{1-\rho^2}Z$

Joint density

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right\}$$

Conditionals

$$(Y \mid X = x) \sim \mathcal{N}\left(\rho x, 1-\rho^2\right) \qquad \text{and} \qquad (X \mid Y = y) \sim \mathcal{N}\left(\rho y, 1-\rho^2\right)$$

Independence

$$X \perp\!\!\!\perp Y \iff \rho = 0$$

9.2 Bivariate Normal

Let $X \sim \mathcal{N}\left(\mu_x, \sigma_x^2\right)$ and $Y \sim \mathcal{N}\left(\mu_y, \sigma_y^2\right)$.

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{z}{2(1-\rho^2)}\right\}$$

$$z = \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]$$

Conditional mean and variance

$$\mathbb{E}\left[X \mid Y\right] = \mathbb{E}\left[X\right] + \rho\frac{\sigma_X}{\sigma_Y}(Y - \mathbb{E}\left[Y\right])$$

$$\mathbb{V}\left[X \mid Y\right] = \sigma_X\sqrt{1-\rho^2}$$

9.3 Multivariate Normal

Covariance matrix Σ (Precision matrix Σ^{-1})

$$\Sigma = \begin{pmatrix} \mathbb{V}\left[X_1\right] & \cdots & \mathrm{Cov}\left[X_1, X_k\right] \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}\left[X_k, X_1\right] & \cdots & \mathbb{V}\left[X_k\right] \end{pmatrix}$$

If $X \sim \mathcal{N}(\mu, \Sigma)$,

$$f_X(x) = (2\pi)^{-n/2}\left|\Sigma\right|^{-1/2}\exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$

Properties

- $Z \sim \mathcal{N}(0,1) \wedge X = \mu + \Sigma^{1/2}Z \implies X \sim \mathcal{N}(\mu, \Sigma)$
- $X \sim \mathcal{N}(\mu, \Sigma) \implies \Sigma^{-1/2}(X-\mu) \sim \mathcal{N}(0,1)$
- $X \sim \mathcal{N}(\mu, \Sigma) \implies AX \sim \mathcal{N}\left(A\mu, A\Sigma A^T\right)$
- $X \sim \mathcal{N}(\mu, \Sigma) \wedge \|a\| = k \implies a^TX \sim \mathcal{N}\left(a^T\mu, a^T\Sigma a\right)$

10 Convergence

Let $\{X_1, X_2, \ldots\}$ be a sequence of RV's and let X be another RV. Let F_n denote the CDF of X_n and let F denote the CDF of X.
Types of Convergence

1. In distribution (weakly, in law): $X_n \xrightarrow{\text{D}} X$

$$\lim_{n\to\infty} F_n(t) = F(t) \qquad \forall t \text{ where } F \text{ continuous}$$