- ARMA (p,q), $\phi(B)x_t = \theta(B)w_t$:

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}$$

where $\phi(z) = 1 - \sum_{k=1}^{p} \phi_k z^k$ and $\theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k$

Discrete Fourier Transform (DFT)

$$d(\omega_j) = n^{-1/2} \sum_{i=1}^{n} x_t e^{-2\pi i\omega_j t}$$

Fourier/Fundamental frequencies

$$\omega_j = j/n$$

Inverse DFT

$$x_t = n^{-1/2} \sum_{j=0}^{n-1} d(\omega_j) e^{2\pi i\omega_j t}$$

Periodogram

$$I(j/n) = |d(j/n)|^2$$

Scaled Periodogram

$$\begin{aligned} P(j/n) &= \frac{4}{n} I(j/n) \\ &= \left(\frac{2}{n}\sum_{t=1}^{n} x_t \cos(2\pi tj/n)\right)^2 + \left(\frac{2}{n}\sum_{t=1}^{n} x_t \sin(2\pi tj/n)\right)^2 \end{aligned}$$

22 Math

22.1 Gamma Function

- Ordinary: $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$
- Upper incomplete: $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt$
- Lower incomplete: $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \alpha > 1$
- $\Gamma(n) = (n-1)! \qquad n \in \mathbb{N}$
- $\Gamma(0) = \Gamma(-1) = \infty$
- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(-1/2) = -2\Gamma(1/2)$

22.2 Beta Function

- Ordinary: $\mathrm{B}(x,y) = \mathrm{B}(y,x) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
- Incomplete: $\mathrm{B}(x;\,a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$
- Regularized incomplete:
  $I_x(a,b) = \frac{\mathrm{B}(x;\,a,b)}{\mathrm{B}(a,b)} \overset{a,b\in\mathbb{N}}{=} \sum_{j=a}^{a+b-1} \frac{(a+b-1)!}{j!(a+b-1-j)!} x^j (1-x)^{a+b-1-j}$
- $I_0(a,b) = 0 \qquad I_1(a,b) = 1$
- $I_x(a,b) = 1 - I_{1-x}(b,a)$

22.3 Series

Finite

- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} (2k-1) = n^2$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\sum_{k=0}^{n} c^k = \frac{c^{n+1}-1}{c-1} \qquad c \neq 1$

Binomial

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- $\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}$
- $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$
- Vandermonde's Identity:
  $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$
- Binomial Theorem:
  $\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n$

Infinite

- $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}, \quad \sum_{k=1}^{\infty} p^k = \frac{p}{1-p} \qquad |p| < 1$
- $\sum_{k=0}^{\infty} kp^{k-1} = \frac{d}{dp}\left(\sum_{k=0}^{\infty} p^k\right) = \frac{d}{dp}\left(\frac{1}{1-p}\right) = \frac{1}{(1-p)^2} \qquad |p| < 1$
- $\sum_{k=0}^{\infty} \binom{r+k-1}{k} x^k = (1-x)^{-r} \qquad r \in \mathbb{N}^+$
- $\sum_{k=0}^{\infty} \binom{\alpha}{k} p^k = (1+p)^\alpha \qquad |p| < 1\,,\ \alpha \in \mathbb{C}$

22.4 Combinatorics

Sampling

k out of n	w/o replacement	w/ replacement
ordered	$n^{\underline{k}} = \prod_{i=0}^{k-1}(n-i) = \frac{n!}{(n-k)!}$	n^k
unordered	$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$	$\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$

Stirling numbers, 2^{nd} kind

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \qquad 1 \le k \le n \qquad \left\{ {n \atop 0} \right\} = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

Partitions

$$P_{n+k,k} = \sum_{i=1}^{n} P_{n,i} \qquad k > n:\ P_{n,k} = 0 \qquad n \ge 1:\ P_{n,0} = 0,\ P_{0,0} = 1$$

Balls and Urns $\quad f : B \to U \quad D$ = distinguishable, $\neg D$ = indistinguishable.

$\lvert B\rvert = n$, $\lvert U\rvert = m$	f arbitrary	f injective	f surjective	f bijective
$B : D,\ U : D$	m^n	$\begin{cases} m^{\underline{n}} & m \ge n \\ 0 & \text{else} \end{cases}$	$m! \left\{ {n \atop m} \right\}$	$\begin{cases} n! & m = n \\ 0 & \text{else} \end{cases}$
$B : \neg D,\ U : D$	$\binom{m+n-1}{n}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$	$\begin{cases} 1 & m = n \\ 0 & \text{else} \end{cases}$
$B : D,\ U : \neg D$	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & m \ge n \\ 0 & \text{else} \end{cases}$	$\left\{ {n \atop m} \right\}$	$\begin{cases} 1 & m = n \\ 0 & \text{else} \end{cases}$
$B : \neg D,\ U : \neg D$	$\sum_{k=1}^{m} P_{n,k}$	$\begin{cases} 1 & m \ge n \\ 0 & \text{else} \end{cases}$	$P_{n,m}$	$\begin{cases} 1 & m = n \\ 0 & \text{else} \end{cases}$

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