Nadaraya-Watson Kernel Estimator

$$\widehat{r}(x) = \sum_{i=1}^{n} w_i(x) Y_i$$

$$w_i(x) = \frac{K\left(\frac{x-x_i}{h}\right)}{\sum_{j=1}^{n} K\left(\frac{x-x_j}{h}\right)} \qquad \in [0,1]$$

$$R(\widehat{r}_n, r) \approx \frac{h^4}{4}\left(\int x^2 K^2(x)\,dx\right)^4 \int \left(r''(x) + 2r'(x)\frac{f'(x)}{f(x)}\right)^2 dx + \int \frac{\sigma^2 \int K^2(x)\,dx}{nhf(x)}\,dx$$

$$h^* \approx \frac{c_1}{n^{1/5}}$$

$$R^*(\widehat{r}_n, r) \approx \frac{c_2}{n^{4/5}}$$

Cross-validation estimate of $\mathbb{E}\left[J(h)\right]$

$$\widehat{J}_{CV}(h) = \sum_{i=1}^{n}(Y_i - \widehat{r}_{(-i)}(x_i))^2 = \sum_{i=1}^{n} \frac{(Y_i - \widehat{r}(x_i))^2}{\left(1 - \frac{K(0)}{\sum_{j=1}^{n} K\left(\frac{x-x_j}{h}\right)}\right)^2}$$

19.3 Smoothing Using Orthogonal Functions

Approximation

$$r(x) = \sum_{j=1}^{\infty} \beta_j \phi_j(x) \approx \sum_{j=1}^{J} \beta_j \phi_j(x)$$

Multivariate regression

$$Y = \Phi\beta + \eta$$

$$\text{where} \quad \eta_i = \epsilon_i \quad \text{and} \quad \Phi = \begin{pmatrix} \phi_0(x_1) & \cdots & \phi_J(x_1) \\ \vdots & \ddots & \vdots \\ \phi_0(x_n) & \cdots & \phi_J(x_n) \end{pmatrix}$$

Least squares estimator

$$\widehat{\beta} = (\Phi^T\Phi)^{-1}\Phi^T Y$$
$$\approx \frac{1}{n}\Phi^T Y \quad \text{(for equally spaced observations only)}$$

Cross-validation estimate of $\mathbb{E}\left[J(h)\right]$

$$\widehat{R}_{CV}(J) = \sum_{i=1}^{n}\left(Y_i - \sum_{j=1}^{J} \phi_j(x_i)\widehat{\beta}_{j,(-i)}\right)^2$$

20 Stochastic Processes

Stochastic Process

$$\{X_t : t \in T\} \qquad T = \begin{cases} \{0, \pm 1, \dots\} = \mathbb{Z} & \text{discrete} \\ [0, \infty) & \text{continuous} \end{cases}$$

- Notations X_t, $X(t)$
- State space $\mathcal{X}$
- Index set T

20.1 Markov Chains

Markov chain

$$\mathbb{P}\left[X_n = x \mid X_0, \dots, X_{n-1}\right] = \mathbb{P}\left[X_n = x \mid X_{n-1}\right] \quad \forall n \in T, x \in \mathcal{X}$$

Transition probabilities

$$p_{ij} \equiv \mathbb{P}\left[X_{n+1} = j \mid X_n = i\right]$$
$$p_{ij}(n) \equiv \mathbb{P}\left[X_{m+n} = j \mid X_m = i\right] \quad \text{n-step}$$

Transition matrix $\mathbf{P}$ (n-step: $\mathbf{P}_n$)

- (i,j) element is p_{ij}
- $p_{ij} > 0$
- $\sum_i p_{ij} = 1$

Chapman-Kolmogorov

$$p_{ij}(m+n) = \sum_k p_{ij}(m) p_{kj}(n)$$

$$\mathbf{P}_{m+n} = \mathbf{P}_m \mathbf{P}_n$$

$$\mathbf{P}_n = \mathbf{P} \times \cdots \times \mathbf{P} = \mathbf{P}^n$$

Marginal probability

$$\mu_n = (\mu_n(1), \dots, \mu_n(N)) \quad \text{where} \quad \mu_i(i) = \mathbb{P}\left[X_n = i\right]$$
$$\mu_0 \triangleq \text{initial distribution}$$
$$\mu_n = \mu_0 \mathbf{P}^n$$

20.2 Poisson Processes

Poisson process

- $\{X_t : t \in [0,\infty)\}$ = number of events up to and including time t
- $X_0 = 0$
- Independent increments:

$$\forall t_0 < \cdots < t_n : X_{t_1} - X_{t_0} \perp\!\!\!\perp \cdots \perp\!\!\!\perp X_{t_n} - X_{t_{n-1}}$$

- Intensity function $\lambda(t)$
  - $\mathbb{P}\left[X_{t+h} - X_t = 1\right] = \lambda(t)h + o(h)$
  - $\mathbb{P}\left[X_{t+h} - X_t = 2\right] = o(h)$
- $X_{s+t} - X_s \sim \text{Po}\left(m(s+t) - m(s)\right)$ where $m(t) = \int_0^t \lambda(s)\,ds$

Homogeneous Poisson process

$$\lambda(t) \equiv \lambda \implies X_t \sim \text{Po}\left(\lambda t\right) \qquad \lambda > 0$$

Waiting times

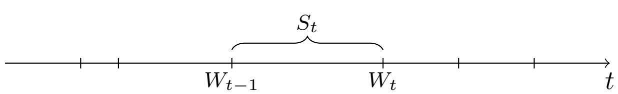
$$W_t := \text{time at which } X_t \text{ occurs}$$

$$W_t \sim \text{Gamma}\left(t, \frac{1}{\lambda}\right)$$

Interarrival times

$$S_t = W_{t+1} - W_t$$

$$S_t \sim \text{Exp}\left(\frac{1}{\lambda}\right)$$



21 Time Series

Mean function

$$\mu_{x_t} = \mathbb{E}\left[x_t\right] = \int_{-\infty}^{\infty} x f_t(x)\,dx$$

Autocovariance function

$$\gamma_x(s,t) = \mathbb{E}\left[(x_s - \mu_s)(x_t - \mu_t)\right] = \mathbb{E}\left[x_s x_t\right] - \mu_s \mu_t$$

$$\gamma_x(t,t) = \mathbb{E}\left[(x_t - \mu_t)^2\right] = \mathbb{V}\left[x_t\right]$$

Autocorrelation function (ACF)

$$\rho(s,t) = \frac{\text{Cov}\left[x_s, x_t\right]}{\sqrt{\mathbb{V}\left[x_s\right]\mathbb{V}\left[x_t\right]}} = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

Cross-covariance function (CCV)

$$\gamma_{xy}(s,t) = \mathbb{E}\left[(x_s - \mu_{x_s})(y_t - \mu_{y_t})\right]$$

Cross-correlation function (CCF)

$$\rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}}$$

Backshift operator

$$B^k(x_t) = x_{t-k}$$

Difference operator

$$\nabla^d = (1 - B)^d$$

White noise

- $w_t \sim wn(0, \sigma_w^2)$
- Gaussian: $w_t \stackrel{iid}{\sim} \mathcal{N}\left(0, \sigma_w^2\right)$
- $\mathbb{E}\left[w_t\right] = 0 \quad t \in T$
- $\mathbb{V}\left[w_t\right] = \sigma^2 \quad t \in T$
- $\gamma_w(s,t) = 0 \quad s \neq t \ \wedge\ s,t \in T$

Random walk

- Drift δ
- $x_t = \delta t + \sum_{j=1}^t w_j$
- $\mathbb{E}\left[x_t\right] = \delta t$

Symmetric moving average

$$m_t = \sum_{j=-k}^{k} a_j x_{t-j} \qquad \text{where } a_j = a_{-j} \geq 0 \text{ and } \sum_{j=-k}^{k} a_j = 1$$