32 String Matching

Text-editing programs frequently need to find all occurrences of a pattern in the text. Typically, the text is a document being edited, and the pattern searched for is a particular word supplied by the user. Efficient algorithms for this problem—called "string matching"—can greatly aid the responsiveness of the text-editing program. Among their many other applications, string-matching algorithms search for particular patterns in DNA sequences. Internet search engines also use them to find Web pages relevant to queries.

We formalize the string-matching problem as follows. We assume that the text is an array $T[1\,..\,n]$ of length n and that the pattern is an array $P[1\,..\,m]$ of length $m \le n$. We further assume that the elements of P and T are characters drawn from a finite alphabet Σ. For example, we may have $\Sigma = \{\texttt{0},\texttt{1}\}$ or $\Sigma = \{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$. The character arrays P and T are often called ***strings*** of characters.

Referring to Figure 32.1, we say that pattern P ***occurs with shift s*** in text T (or, equivalently, that pattern P ***occurs beginning at position s + 1*** in text T) if $0 \le s \le n - m$ and $T[s+1\,..\,s+m] = P[1\,..\,m]$ (that is, if $T[s+j] = P[j]$, for $1 \le j \le m$). If P occurs with shift s in T, then we call s a ***valid shift***; otherwise, we call s an ***invalid shift***. The ***string-matching problem*** is the problem of finding all valid shifts with which a given pattern P occurs in a given text T.

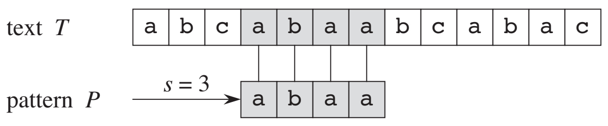


Figure 32.1 An example of the string-matching problem, where we want to find all occurrences of the pattern $P = \texttt{abaa}$ in the text $T = \texttt{abcabaabcabac}$. The pattern occurs only once in the text, at shift $s = 3$, which we call a valid shift. A vertical line connects each character of the pattern to its matching character in the text, and all matched characters are shaded.

Algorithm	Preprocessing time	Matching time
Naive	0	$O((n-m+1)m)$
Rabin-Karp	$\Theta(m)$	$O((n-m+1)m)$
Finite automaton	$O(m\,\lvert\Sigma\rvert)$	$\Theta(n)$
Knuth-Morris-Pratt	$\Theta(m)$	$\Theta(n)$

Figure 32.2 The string-matching algorithms in this chapter and their preprocessing and matching times.

Except for the naive brute-force algorithm, which we review in Section 32.1, each string-matching algorithm in this chapter performs some preprocessing based on the pattern and then finds all valid shifts; we call this latter phase "matching." Figure 32.2 shows the preprocessing and matching times for each of the algorithms in this chapter. The total running time of each algorithm is the sum of the preprocessing and matching times. Section 32.2 presents an interesting string-matching algorithm, due to Rabin and Karp. Although the $\Theta((n-m+1)m)$ worst-case running time of this algorithm is no better than that of the naive method, it works much better on average and in practice. It also generalizes nicely to other pattern-matching problems. Section 32.3 then describes a string-matching algorithm that begins by constructing a finite automaton specifically designed to search for occurrences of the given pattern P in a text. This algorithm takes $O(m\,|\Sigma|)$ preprocessing time, but only $\Theta(n)$ matching time. Section 32.4 presents the similar, but much cleverer, Knuth-Morris-Pratt (or KMP) algorithm; it has the same $\Theta(n)$ matching time, and it reduces the preprocessing time to only $\Theta(m)$.

Notation and terminology

We denote by Σ^* (read "sigma-star") the set of all finite-length strings formed using characters from the alphabet Σ. In this chapter, we consider only finite-length strings. The zero-length ***empty string***, denoted ε, also belongs to Σ^*. The length of a string x is denoted $|x|$. The ***concatenation*** of two strings x and y, denoted xy, has length $|x| + |y|$ and consists of the characters from x followed by the characters from y.

We say that a string w is a ***prefix*** of a string x, denoted $w \sqsubset x$, if $x = wy$ for some string $y \in \Sigma^*$. Note that if $w \sqsubset x$, then $|w| \le |x|$. Similarly, we say that a string w is a ***suffix*** of a string x, denoted $w \sqsupset x$, if $x = yw$ for some $y \in \Sigma^*$. As with a prefix, $w \sqsupset x$ implies $|w| \le |x|$. For example, we have $\texttt{ab} \sqsubset \texttt{abcca}$ and $\texttt{cca} \sqsupset \texttt{abcca}$. The empty string ε is both a suffix and a prefix of every string. For any strings x and y and any character a, we have $x \sqsupset y$ if and only if $xa \sqsupset ya$.

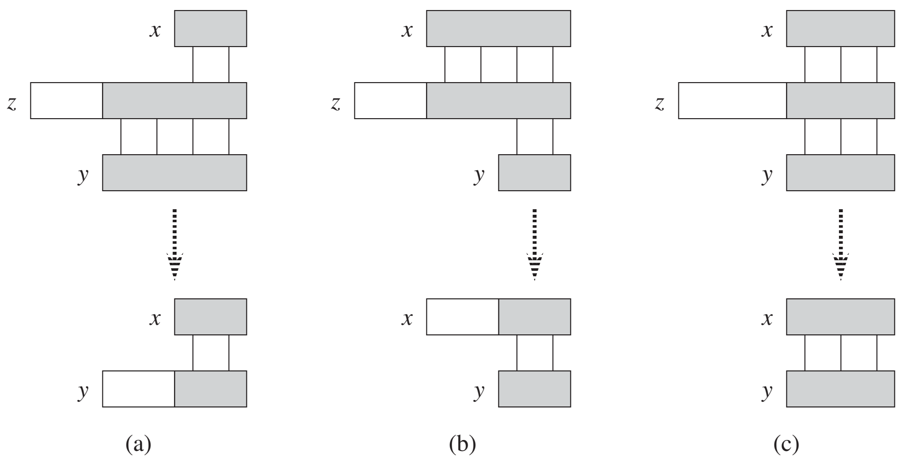


Figure 32.3 A graphical proof of Lemma 32.1. We suppose that $x \sqsupset z$ and $y \sqsupset z$. The three parts of the figure illustrate the three cases of the lemma. Vertical lines connect matching regions (shown shaded) of the strings. **(a)** If $|x| \le |y|$, then $x \sqsupset y$. **(b)** If $|x| \ge |y|$, then $y \sqsupset x$. **(c)** If $|x| = |y|$, then $x = y$.

Also note that $\sqsubset$ and $\sqsupset$ are transitive relations. The following lemma will be useful later.

Lemma 32.1 (Overlapping-suffix lemma)
Suppose that x, y, and z are strings such that $x \sqsupset z$ and $y \sqsupset z$. If $|x| \le |y|$, then $x \sqsupset y$. If $|x| \ge |y|$, then $y \sqsupset x$. If $|x| = |y|$, then $x = y$.

Proof See Figure 32.3 for a graphical proof. ■

For brevity of notation, we denote the k-character prefix $P[1\,..\,k]$ of the pattern $P[1\,..\,m]$ by P_k. Thus, $P_0 = \varepsilon$ and $P_m = P = P[1\,..\,m]$. Similarly, we denote the k-character prefix of the text T by T_k. Using this notation, we can state the string-matching problem as that of finding all shifts s in the range $0 \le s \le n - m$ such that $P \sqsupset T_{s+m}$.

In our pseudocode, we allow two equal-length strings to be compared for equality as a primitive operation. If the strings are compared from left to right and the comparison stops when a mismatch is discovered, we assume that the time taken by such a test is a linear function of the number of matching characters discovered. To be precise, the test "$x == y$" is assumed to take time $\Theta(t + 1)$, where t is the length of the longest string z such that $z \sqsubset x$ and $z \sqsubset y$. (We write $\Theta(t + 1)$ rather than $\Theta(t)$ to handle the case in which $t = 0$; the first characters compared do not match, but it takes a positive amount of time to perform this comparison.)

32.1 The naive string-matching algorithm

The naive algorithm finds all valid shifts using a loop that checks the condition $P[1..m] = T[s+1..s+m]$ for each of the $n-m+1$ possible values of s.

```
NAIVE-STRING-MATCHER(T, P)
1  $n = T.length$
2  $m = P.length$
3  for $s = 0$ to $n - m$
4      if $P[1..m] == T[s+1..s+m]$
5          print "Pattern occurs with shift" $s$
```

Figure 32.4 portrays the naive string-matching procedure as sliding a "template" containing the pattern over the text, noting for which shifts all of the characters on the template equal the corresponding characters in the text. The **for** loop of lines 3–5 considers each possible shift explicitly. The test in line 4 determines whether the current shift is valid; this test implicitly loops to check corresponding character positions until all positions match successfully or a mismatch is found. Line 5 prints out each valid shift s.

Procedure NAIVE-STRING-MATCHER takes time $O((n-m+1)m)$, and this bound is tight in the worst case. For example, consider the text string $\texttt{a}^n$ (a string of n a's) and the pattern $\texttt{a}^m$. For each of the $n-m+1$ possible values of the shift s, the implicit loop on line 4 to compare corresponding characters must execute m times to validate the shift. The worst-case running time is thus $\Theta((n-m+1)m)$, which is $\Theta(n^2)$ if $m = \lfloor n/2 \rfloor$. Because it requires no preprocessing, NAIVE-STRING-MATCHER's running time equals its matching time.

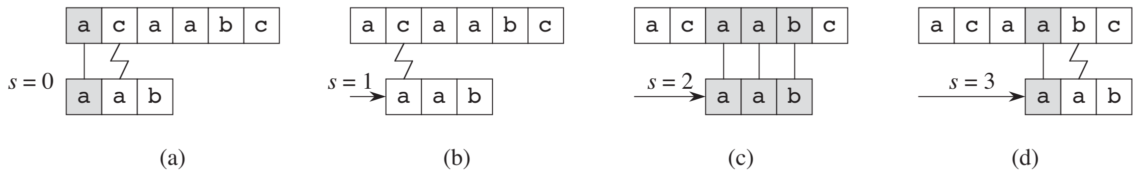


Figure 32.4 The operation of the naive string matcher for the pattern $P = \texttt{aab}$ and the text $T = \texttt{acaabc}$. We can imagine the pattern P as a template that we slide next to the text. **(a)–(d)** The four successive alignments tried by the naive string matcher. In each part, vertical lines connect corresponding regions found to match (shown shaded), and a jagged line connects the first mismatched character found, if any. The algorithm finds one occurrence of the pattern, at shift $s = 2$, shown in part (c).

As we shall see, NAIVE-STRING-MATCHER is not an optimal procedure for this problem. Indeed, in this chapter we shall see that the Knuth-Morris-Pratt algorithm is much better in the worst case. The naive string-matcher is inefficient because it entirely ignores information gained about the text for one value of s when it considers other values of s. Such information can be quite valuable, however. For example, if $P = \texttt{aaab}$ and we find that $s = 0$ is valid, then none of the shifts 1, 2, or 3 are valid, since $T[4] = \texttt{b}$. In the following sections, we examine several ways to make effective use of this sort of information.

Exercises

32.1-1
Show the comparisons the naive string matcher makes for the pattern $P = \texttt{0001}$ in the text $T = \texttt{000010001010001}$.

32.1-2
Suppose that all characters in the pattern P are different. Show how to accelerate NAIVE-STRING-MATCHER to run in time $O(n)$ on an n-character text T.

32.1-3
Suppose that pattern P and text T are *randomly* chosen strings of length m and n, respectively, from the d-ary alphabet $\Sigma_d = \{0, 1, \ldots, d-1\}$, where $d \geq 2$. Show that the *expected* number of character-to-character comparisons made by the implicit loop in line 4 of the naive algorithm is

$$(n - m + 1)\frac{1 - d^{-m}}{1 - d^{-1}} \leq 2(n - m + 1)$$

over all executions of this loop. (Assume that the naive algorithm stops comparing characters for a given shift once it finds a mismatch or matches the entire pattern.) Thus, for randomly chosen strings, the naive algorithm is quite efficient.

32.1-4
Suppose we allow the pattern P to contain occurrences of a ***gap character*** $\diamond$ that can match an *arbitrary* string of characters (even one of zero length). For example, the pattern $\texttt{ab}\diamond\texttt{ba}\diamond\texttt{c}$ occurs in the text $\texttt{cabccbacbacab}$ as

$$\texttt{c}\;\underbrace{\texttt{ab}}_{\texttt{ab}}\;\underbrace{\texttt{cc}}_{\diamond}\;\underbrace{\texttt{ba}}_{\texttt{ba}}\;\underbrace{\texttt{cba}}_{\diamond}\;\underbrace{\texttt{c}}_{\texttt{c}}\;\texttt{ab}$$

and as

$$\texttt{c}\;\underbrace{\texttt{ab}}_{\texttt{ab}}\;\underbrace{\texttt{ccbac}}_{\diamond}\;\underbrace{\texttt{ba}}_{\texttt{ba}}\;\underbrace{\quad}_{\diamond}\;\underbrace{\texttt{c}}_{\texttt{c}}\;\texttt{ab}\,.$$

Note that the gap character may occur an arbitrary number of times in the pattern but not at all in the text. Give a polynomial-time algorithm to determine whether such a pattern P occurs in a given text T, and analyze the running time of your algorithm.

32.2 The Rabin-Karp algorithm

Rabin and Karp proposed a string-matching algorithm that performs well in practice and that also generalizes to other algorithms for related problems, such as two-dimensional pattern matching. The Rabin-Karp algorithm uses $\Theta(m)$ preprocessing time, and its worst-case running time is $\Theta((n-m+1)m)$. Based on certain assumptions, however, its average-case running time is better.

This algorithm makes use of elementary number-theoretic notions such as the equivalence of two numbers modulo a third number. You might want to refer to Section 31.1 for the relevant definitions.

For expository purposes, let us assume that $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, \mathtt{9}\}$, so that each character is a decimal digit. (In the general case, we can assume that each character is a digit in radix-d notation, where $d = |\Sigma|$.) We can then view a string of k consecutive characters as representing a length-k decimal number. The character string `31415` thus corresponds to the decimal number 31,415. Because we interpret the input characters as both graphical symbols and digits, we find it convenient in this section to denote them as we would digits, in our standard text font.

Given a pattern $P[1\,..\,m]$, let p denote its corresponding decimal value. In a similar manner, given a text $T[1\,..\,n]$, let t_s denote the decimal value of the length-m substring $T[s+1\,..\,s+m]$, for $s = 0, 1, \ldots, n-m$. Certainly, $t_s = p$ if and only if $T[s+1\,..\,s+m] = P[1\,..\,m]$; thus, s is a valid shift if and only if $t_s = p$. If we could compute p in time $\Theta(m)$ and all the t_s values in a total of $\Theta(n-m+1)$ time,[1] then we could determine all valid shifts s in time $\Theta(m) + \Theta(n-m+1) = \Theta(n)$ by comparing p with each of the t_s values. (For the moment, let's not worry about the possibility that p and the t_s values might be very large numbers.)

We can compute p in time $\Theta(m)$ using Horner's rule (see Section 30.1):

$$p = P[m] + 10\,(P[m-1] + 10(P[m-2] + \cdots + 10(P[2] + 10P[1])\cdots))\ .$$

Similarly, we can compute t_0 from $T[1\,..\,m]$ in time $\Theta(m)$.

[1] We write $\Theta(n-m+1)$ instead of $\Theta(n-m)$ because s takes on $n-m+1$ different values. The "+1" is significant in an asymptotic sense because when $m = n$, computing the lone t_s value takes $\Theta(1)$ time, not $\Theta(0)$ time.

To compute the remaining values $t_1, t_2, \ldots, t_{n-m}$ in time $\Theta(n-m)$, we observe that we can compute t_{s+1} from t_s in constant time, since

$$t_{s+1} = 10(t_s - 10^{m-1}T[s+1]) + T[s+m+1] \, . \tag{32.1}$$

Subtracting $10^{m-1}T[s+1]$ removes the high-order digit from t_s, multiplying the result by 10 shifts the number left by one digit position, and adding $T[s+m+1]$ brings in the appropriate low-order digit. For example, if $m = 5$ and $t_s = 31415$, then we wish to remove the high-order digit $T[s+1] = 3$ and bring in the new low-order digit (suppose it is $T[s+5+1] = 2$) to obtain

$$\begin{aligned} t_{s+1} &= 10(31415 - 10000 \cdot 3) + 2 \\ &= 14152 \, . \end{aligned}$$

If we precompute the constant 10^{m-1} (which we can do in time $O(\lg m)$ using the techniques of Section 31.6, although for this application a straightforward $O(m)$-time method suffices), then each execution of equation (32.1) takes a constant number of arithmetic operations. Thus, we can compute p in time $\Theta(m)$, and we can compute all of $t_0, t_1, \ldots, t_{n-m}$ in time $\Theta(n-m+1)$. Therefore, we can find all occurrences of the pattern $P[1\,..\,m]$ in the text $T[1\,..\,n]$ with $\Theta(m)$ preprocessing time and $\Theta(n-m+1)$ matching time.

Until now, we have intentionally overlooked one problem: p and t_s may be too large to work with conveniently. If P contains m characters, then we cannot reasonably assume that each arithmetic operation on p (which is m digits long) takes "constant time." Fortunately, we can solve this problem easily, as Figure 32.5 shows: compute p and the t_s values modulo a suitable modulus q. We can compute p modulo q in $\Theta(m)$ time and all the t_s values modulo q in $\Theta(n-m+1)$ time. If we choose the modulus q as a prime such that $10q$ just fits within one computer word, then we can perform all the necessary computations with single-precision arithmetic. In general, with a d-ary alphabet $\{0, 1, \ldots, d-1\}$, we choose q so that dq fits within a computer word and adjust the recurrence equation (32.1) to work modulo q, so that it becomes

$$t_{s+1} = (d(t_s - T[s+1]h) + T[s+m+1]) \bmod q \, , \tag{32.2}$$

where $h \equiv d^{m-1} \pmod{q}$ is the value of the digit "1" in the high-order position of an m-digit text window.

The solution of working modulo q is not perfect, however: $t_s \equiv p \pmod{q}$ does not imply that $t_s = p$. On the other hand, if $t_s \not\equiv p \pmod{q}$, then we definitely have that $t_s \neq p$, so that shift s is invalid. We can thus use the test $t_s \equiv p \pmod{q}$ as a fast heuristic test to rule out invalid shifts s. Any shift s for which $t_s \equiv p \pmod{q}$ must be tested further to see whether s is really valid or we just have a ***spurious hit***. This additional test explicitly checks the condition

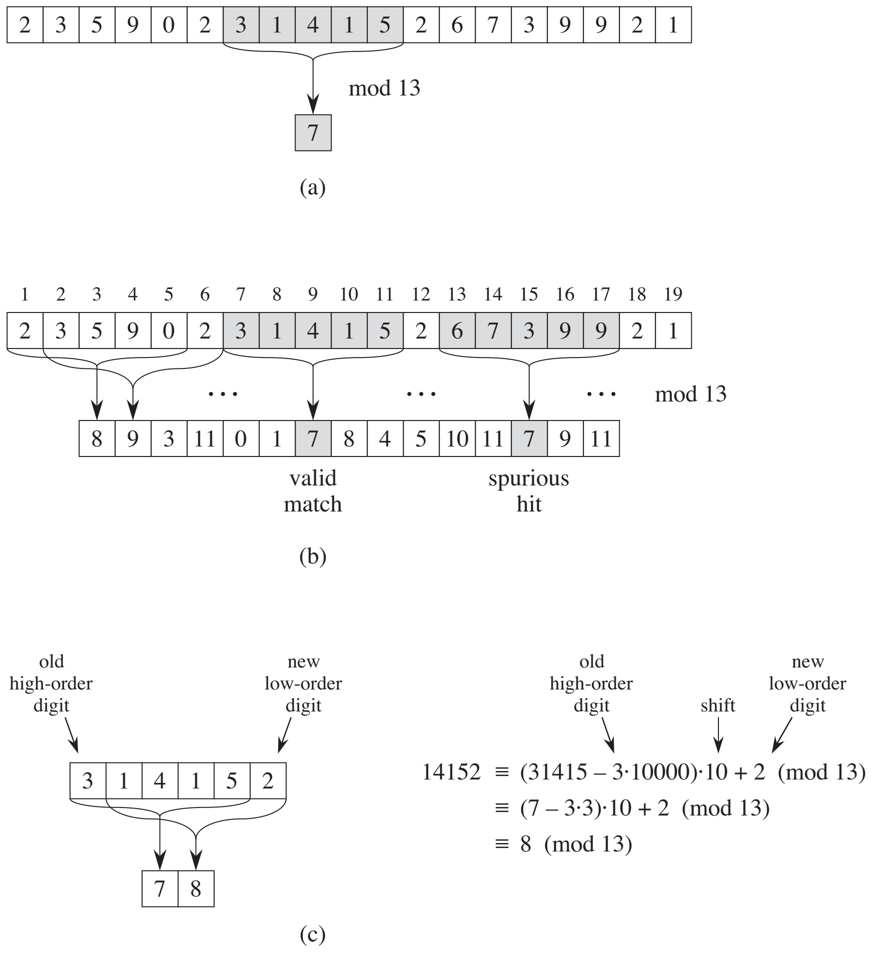


Figure 32.5 The Rabin-Karp algorithm. Each character is a decimal digit, and we compute values modulo 13. **(a)** A text string. A window of length 5 is shown shaded. The numerical value of the shaded number, computed modulo 13, yields the value 7. **(b)** The same text string with values computed modulo 13 for each possible position of a length-5 window. Assuming the pattern $P = 31415$, we look for windows whose value modulo 13 is 7, since $31415 \equiv 7 \pmod{13}$. The algorithm finds two such windows, shown shaded in the figure. The first, beginning at text position 7, is indeed an occurrence of the pattern, while the second, beginning at text position 13, is a spurious hit. **(c)** How to compute the value for a window in constant time, given the value for the previous window. The first window has value 31415. Dropping the high-order digit 3, shifting left (multiplying by 10), and then adding in the low-order digit 2 gives us the new value 14152. Because all computations are performed modulo 13, the value for the first window is 7, and the value for the new window is 8.

$P[1 \mathinner{.\,.} m] = T[s+1 \mathinner{.\,.} s+m]$. If q is large enough, then we hope that spurious hits occur infrequently enough that the cost of the extra checking is low.

The following procedure makes these ideas precise. The inputs to the procedure are the text T, the pattern P, the radix d to use (which is typically taken to be $|\Sigma|$), and the prime q to use.

```
RABIN-KARP-MATCHER(T, P, d, q)
 1  n = T.length
 2  m = P.length
 3  h = d^{m-1} mod q
 4  p = 0
 5  t_0 = 0
 6  for i = 1 to m                  // preprocessing
 7      p = (dp + P[i]) mod q
 8      t_0 = (dt_0 + T[i]) mod q
 9  for s = 0 to n - m              // matching
10      if p == t_s
11          if P[1..m] == T[s+1..s+m]
12              print "Pattern occurs with shift" s
13      if s < n - m
14          t_{s+1} = (d(t_s - T[s+1]h) + T[s+m+1]) mod q
```

The procedure RABIN-KARP-MATCHER works as follows. All characters are interpreted as radix-d digits. The subscripts on t are provided only for clarity; the program works correctly if all the subscripts are dropped. Line 3 initializes h to the value of the high-order digit position of an m-digit window. Lines 4–8 compute p as the value of $P[1 \mathinner{.\,.} m] \bmod q$ and t_0 as the value of $T[1 \mathinner{.\,.} m] \bmod q$. The **for** loop of lines 9–14 iterates through all possible shifts s, maintaining the following invariant:

Whenever line 10 is executed, $t_s = T[s+1 \mathinner{.\,.} s+m] \bmod q$.

If $p = t_s$ in line 10 (a "hit"), then line 11 checks to see whether $P[1 \mathinner{.\,.} m] = T[s+1 \mathinner{.\,.} s+m]$ in order to rule out the possibility of a spurious hit. Line 12 prints out any valid shifts that are found. If $s < n - m$ (checked in line 13), then the **for** loop will execute at least one more time, and so line 14 first executes to ensure that the loop invariant holds when we get back to line 10. Line 14 computes the value of $t_{s+1} \bmod q$ from the value of $t_s \bmod q$ in constant time using equation (32.2) directly.

RABIN-KARP-MATCHER takes $\Theta(m)$ preprocessing time, and its matching time is $\Theta((n-m+1)m)$ in the worst case, since (like the naive string-matching algorithm) the Rabin-Karp algorithm explicitly verifies every valid shift. If $P = \mathtt{a}^m$

and $T = \texttt{a}^n$, then verifying takes time $\Theta((n-m+1)m)$, since each of the $n-m+1$ possible shifts is valid.

In many applications, we expect few valid shifts—perhaps some constant c of them. In such applications, the expected matching time of the algorithm is only $O((n-m+1)+cm) = O(n+m)$, plus the time required to process spurious hits. We can base a heuristic analysis on the assumption that reducing values modulo q acts like a random mapping from Σ^* to $\mathbb{Z}_q$. (See the discussion on the use of division for hashing in Section 11.3.1. It is difficult to formalize and prove such an assumption, although one viable approach is to assume that q is chosen randomly from integers of the appropriate size. We shall not pursue this formalization here.) We can then expect that the number of spurious hits is $O(n/q)$, since we can estimate the chance that an arbitrary t_s will be equivalent to p, modulo q, as $1/q$. Since there are $O(n)$ positions at which the test of line 10 fails and we spend $O(m)$ time for each hit, the expected matching time taken by the Rabin-Karp algorithm is

$$O(n) + O(m(v + n/q)) ,$$

where v is the number of valid shifts. This running time is $O(n)$ if $v = O(1)$ and we choose $q \geq m$. That is, if the expected number of valid shifts is small ($O(1)$) and we choose the prime q to be larger than the length of the pattern, then we can expect the Rabin-Karp procedure to use only $O(n+m)$ matching time. Since $m \leq n$, this expected matching time is $O(n)$.

Exercises

32.2-1

Working modulo $q = 11$, how many spurious hits does the Rabin-Karp matcher encounter in the text $T = 3141592653589793$ when looking for the pattern $P = 26$?

32.2-2

How would you extend the Rabin-Karp method to the problem of searching a text string for an occurrence of any one of a given set of k patterns? Start by assuming that all k patterns have the same length. Then generalize your solution to allow the patterns to have different lengths.

32.2-3

Show how to extend the Rabin-Karp method to handle the problem of looking for a given $m \times m$ pattern in an $n \times n$ array of characters. (The pattern may be shifted vertically and horizontally, but it may not be rotated.)

32.2-4

Alice has a copy of a long n-bit file $A = \langle a_{n-1}, a_{n-2}, \ldots, a_0 \rangle$, and Bob similarly has an n-bit file $B = \langle b_{n-1}, b_{n-2}, \ldots, b_0 \rangle$. Alice and Bob wish to know if their files are identical. To avoid transmitting all of A or B, they use the following fast probabilistic check. Together, they select a prime $q > 1000n$ and randomly select an integer x from $\{0, 1, \ldots, q-1\}$. Then, Alice evaluates

$$A(x) = \left(\sum_{i=0}^{n-1} a_i x^i \right) \bmod q$$

and Bob similarly evaluates $B(x)$. Prove that if $A \neq B$, there is at most one chance in 1000 that $A(x) = B(x)$, whereas if the two files are the same, $A(x)$ is necessarily the same as $B(x)$. (*Hint:* See Exercise 31.4-4.)

32.3 String matching with finite automata

Many string-matching algorithms build a finite automaton—a simple machine for processing information—that scans the text string T for all occurrences of the pattern P. This section presents a method for building such an automaton. These string-matching automata are very efficient: they examine each text character *exactly once*, taking constant time per text character. The matching time used—after preprocessing the pattern to build the automaton—is therefore $\Theta(n)$. The time to build the automaton, however, can be large if Σ is large. Section 32.4 describes a clever way around this problem.

We begin this section with the definition of a finite automaton. We then examine a special string-matching automaton and show how to use it to find occurrences of a pattern in a text. Finally, we shall show how to construct the string-matching automaton for a given input pattern.

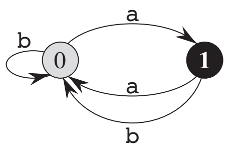
Finite automata

A ***finite automaton*** M, illustrated in Figure 32.6, is a 5-tuple $(Q, q_0, A, \Sigma, \delta)$, where

- Q is a finite set of ***states***,
- $q_0 \in Q$ is the ***start state***,
- $A \subseteq Q$ is a distinguished set of ***accepting states***,
- Σ is a finite ***input alphabet***,
- δ is a function from $Q \times \Sigma$ into Q, called the ***transition function*** of M.

	input	
state	a	b
0	1	0
1	0	0

(a)



(b)

Figure 32.6 A simple two-state finite automaton with state set $Q = \{0, 1\}$, start state $q_0 = 0$, and input alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$. **(a)** A tabular representation of the transition function δ. **(b)** An equivalent state-transition diagram. State 1, shown blackend, is the only accepting state. Directed edges represent transitions. For example, the edge from state 1 to state 0 labeled b indicates that $\delta(1, \texttt{b}) = 0$. This automaton accepts those strings that end in an odd number of a's. More precisely, it accepts a string x if and only if $x = yz$, where $y = \varepsilon$ or y ends with a b, and $z = \texttt{a}^k$, where k is odd. For example, on input abaaa, including the start state, this automaton enters the sequence of states $\langle 0, 1, 0, 1, 0, 1 \rangle$, and so it accepts this input. For input abbaa, it enters the sequence of states $\langle 0, 1, 0, 0, 1, 0 \rangle$, and so it rejects this input.

The finite automaton begins in state q_0 and reads the characters of its input string one at a time. If the automaton is in state q and reads input character a, it moves ("makes a transition") from state q to state $\delta(q, a)$. Whenever its current state q is a member of A, the machine M has ***accepted*** the string read so far. An input that is not accepted is ***rejected***.

A finite automaton M induces a function ϕ, called the ***final-state function***, from Σ^* to Q such that $\phi(w)$ is the state M ends up in after scanning the string w. Thus, M accepts a string w if and only if $\phi(w) \in A$. We define the function ϕ recursively, using the transition function:

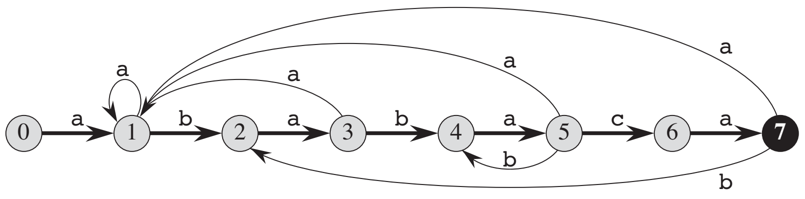
$$\begin{aligned} \phi(\varepsilon) &= q_0 , \\ \phi(wa) &= \delta(\phi(w), a) \quad \text{for } w \in \Sigma^*, a \in \Sigma . \end{aligned}$$

String-matching automata

For a given pattern P, we construct a string-matching automaton in a preprocessing step before using it to search the text string. Figure 32.7 illustrates how we construct the automaton for the pattern $P = \texttt{ababaca}$. From now on, we shall assume that P is a given fixed pattern string; for brevity, we shall not indicate the dependence upon P in our notation.

In order to specify the string-matching automaton corresponding to a given pattern $P[1 \mathinner{..} m]$, we first define an auxiliary function σ, called the ***suffix function*** corresponding to P. The function σ maps Σ^* to $\{0, 1, \ldots, m\}$ such that $\sigma(x)$ is the length of the longest prefix of P that is also a suffix of x:

$$\sigma(x) = \max\{k : P_k \sqsupset x\} . \tag{32.3}$$



(a)

state	input a	input b	input c	P
0	1	0	0	a
1	1	2	0	b
2	3	0	0	a
3	1	4	0	b
4	5	0	0	a
5	1	4	6	c
6	7	0	0	a
7	1	2	0	

(b)

i	—	1	2	3	4	5	6	7	8	9	10	11
$T[i]$	—	a	b	a	b	a	b	a	c	a	b	a
state $\phi(T_i)$	0	1	2	3	4	5	4	5	6	7	2	3

(c)

Figure 32.7 **(a)** A state-transition diagram for the string-matching automaton that accepts all strings ending in the string `ababaca`. State 0 is the start state, and state 7 (shown blackened) is the only accepting state. A directed edge from state i to state j labeled a represents $\delta(i, a) = j$. The right-going edges forming the "spine" of the automaton, shown heavy in the figure, correspond to successful matches between pattern and input characters. The left-going edges correspond to failing matches. Some edges corresponding to failing matches are omitted; by convention, if a state i has no outgoing edge labeled a for some $a \in \Sigma$, then $\delta(i, a) = 0$. **(b)** The corresponding transition function δ, and the pattern string $P =$ `ababaca`. The entries corresponding to successful matches between pattern and input characters are shown shaded. **(c)** The operation of the automaton on the text $T =$ `abababacaba`. Under each text character $T[i]$ appears the state $\phi(T_i)$ that the automaton is in after processing the prefix T_i. The automaton finds one occurrence of the pattern, ending in position 9.

The suffix function σ is well defined since the empty string $P_0 = \varepsilon$ is a suffix of every string. As examples, for the pattern $P =$ `ab`, we have $\sigma(\varepsilon) = 0$, $\sigma($`ccaca`$) = 1$, and $\sigma($`ccab`$) = 2$. For a pattern P of length m, we have $\sigma(x) = m$ if and only if $P \sqsupset x$. From the definition of the suffix function, $x \sqsupset y$ implies $\sigma(x) \leq \sigma(y)$.

We define the string-matching automaton that corresponds to a given pattern $P[1 \,.\,.\, m]$ as follows:

- The state set Q is $\{0, 1, \ldots, m\}$. The start state q_0 is state 0, and state m is the only accepting state.
- The transition function δ is defined by the following equation, for any state q and character a:

$$\delta(q, a) = \sigma(P_q a) . \tag{32.4}$$

We define $\delta(q, a) = \sigma(P_q a)$ because we want to keep track of the longest prefix of the pattern P that has matched the text string T so far. We consider the most recently read characters of T. In order for a substring of T—let's say the substring ending at $T[i]$—to match some prefix P_j of P, this prefix P_j must be a suffix of T_i. Suppose that $q = \phi(T_i)$, so that after reading T_i, the automaton is in state q. We design the transition function δ so that this state number, q, tells us the length of the longest prefix of P that matches a suffix of T_i. That is, in state q, $P_q \sqsupset T_i$ and $q = \sigma(T_i)$. (Whenever $q = m$, all m characters of P match a suffix of T_i, and so we have found a match.) Thus, since $\phi(T_i)$ and $\sigma(T_i)$ both equal q, we shall see (in Theorem 32.4, below) that the automaton maintains the following invariant:

$$\phi(T_i) = \sigma(T_i) . \tag{32.5}$$

If the automaton is in state q and reads the next character $T[i+1] = a$, then we want the transition to lead to the state corresponding to the longest prefix of P that is a suffix of $T_i a$, and that state is $\sigma(T_i a)$. Because P_q is the longest prefix of P that is a suffix of T_i, the longest prefix of P that is a suffix of $T_i a$ is not only $\sigma(T_i a)$, but also $\sigma(P_q a)$. (Lemma 32.3, on page 1000, proves that $\sigma(T_i a) = \sigma(P_q a)$.) Thus, when the automaton is in state q, we want the transition function on character a to take the automaton to state $\sigma(P_q a)$.

There are two cases to consider. In the first case, $a = P[q+1]$, so that the character a continues to match the pattern; in this case, because $\delta(q, a) = q+1$, the transition continues to go along the "spine" of the automaton (the heavy edges in Figure 32.7). In the second case, $a \neq P[q+1]$, so that a does not continue to match the pattern. Here, we must find a smaller prefix of P that is also a suffix of T_i. Because the preprocessing step matches the pattern against itself when creating the string-matching automaton, the transition function quickly identifies the longest such smaller prefix of P.

Let's look at an example. The string-matching automaton of Figure 32.7 has $\delta(5, \texttt{c}) = 6$, illustrating the first case, in which the match continues. To illustrate the second case, observe that the automaton of Figure 32.7 has $\delta(5, \texttt{b}) = 4$. We make this transition because if the automaton reads a `b` in state $q = 5$, then $P_q\texttt{b} = \texttt{ababab}$, and the longest prefix of P that is also a suffix of `ababab` is $P_4 = \texttt{abab}$.

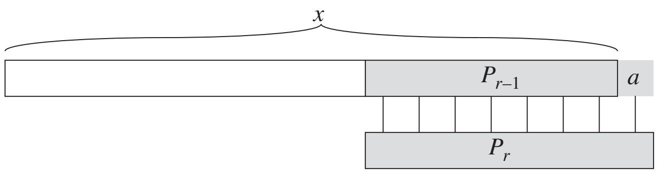


Figure 32.8 An illustration for the proof of Lemma 32.2. The figure shows that $r \le \sigma(x) + 1$, where $r = \sigma(xa)$.

To clarify the operation of a string-matching automaton, we now give a simple, efficient program for simulating the behavior of such an automaton (represented by its transition function δ) in finding occurrences of a pattern P of length m in an input text $T[1 \,..\, n]$. As for any string-matching automaton for a pattern of length m, the state set Q is $\{0, 1, \ldots, m\}$, the start state is 0, and the only accepting state is state m.

```
FINITE-AUTOMATON-MATCHER(T, δ, m)
1  n = T.length
2  q = 0
3  for i = 1 to n
4      q = δ(q, T[i])
5      if q == m
6          print "Pattern occurs with shift" i − m
```

From the simple loop structure of FINITE-AUTOMATON-MATCHER, we can easily see that its matching time on a text string of length n is $\Theta(n)$. This matching time, however, does not include the preprocessing time required to compute the transition function δ. We address this problem later, after first proving that the procedure FINITE-AUTOMATON-MATCHER operates correctly.

Consider how the automaton operates on an input text $T[1 \,..\, n]$. We shall prove that the automaton is in state $\sigma(T_i)$ after scanning character $T[i]$. Since $\sigma(T_i) = m$ if and only if $P \sqsupset T_i$, the machine is in the accepting state m if and only if it has just scanned the pattern P. To prove this result, we make use of the following two lemmas about the suffix function σ.

Lemma 32.2 (Suffix-function inequality)
For any string x and character a, we have $\sigma(xa) \le \sigma(x) + 1$.

Proof Referring to Figure 32.8, let $r = \sigma(xa)$. If $r = 0$, then the conclusion $\sigma(xa) = r \le \sigma(x) + 1$ is trivially satisfied, by the nonnegativity of $\sigma(x)$. Now assume that $r > 0$. Then, $P_r \sqsupset xa$, by the definition of σ. Thus, $P_{r-1} \sqsupset x$, by

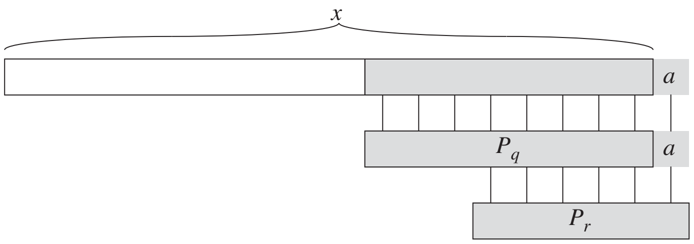


Figure 32.9 An illustration for the proof of Lemma 32.3. The figure shows that $r = \sigma(P_q a)$, where $q = \sigma(x)$ and $r = \sigma(xa)$.

dropping the a from the end of P_r and from the end of xa. Therefore, $r-1 \le \sigma(x)$, since $\sigma(x)$ is the largest k such that $P_k \sqsupset x$, and thus $\sigma(xa) = r \le \sigma(x) + 1$. ■

Lemma 32.3 (Suffix-function recursion lemma)
For any string x and character a, if $q = \sigma(x)$, then $\sigma(xa) = \sigma(P_q a)$.

Proof From the definition of σ, we have $P_q \sqsupset x$. As Figure 32.9 shows, we also have $P_q a \sqsupset xa$. If we let $r = \sigma(xa)$, then $P_r \sqsupset xa$ and, by Lemma 32.2, $r \le q + 1$. Thus, we have $|P_r| = r \le q + 1 = |P_q a|$. Since $P_q a \sqsupset xa$, $P_r \sqsupset xa$, and $|P_r| \le |P_q a|$, Lemma 32.1 implies that $P_r \sqsupset P_q a$. Therefore, $r \le \sigma(P_q a)$, that is, $\sigma(xa) \le \sigma(P_q a)$. But we also have $\sigma(P_q a) \le \sigma(xa)$, since $P_q a \sqsupset xa$. Thus, $\sigma(xa) = \sigma(P_q a)$. ■

We are now ready to prove our main theorem characterizing the behavior of a string-matching automaton on a given input text. As noted above, this theorem shows that the automaton is merely keeping track, at each step, of the longest prefix of the pattern that is a suffix of what has been read so far. In other words, the automaton maintains the invariant (32.5).

Theorem 32.4
If ϕ is the final-state function of a string-matching automaton for a given pattern P and $T[1..n]$ is an input text for the automaton, then

$$\phi(T_i) = \sigma(T_i)$$

for $i = 0, 1, \ldots, n$.

Proof The proof is by induction on i. For $i = 0$, the theorem is trivially true, since $T_0 = \varepsilon$. Thus, $\phi(T_0) = 0 = \sigma(T_0)$.

Now, we assume that $\phi(T_i) = \sigma(T_i)$ and prove that $\phi(T_{i+1}) = \sigma(T_{i+1})$. Let q denote $\phi(T_i)$, and let a denote $T[i+1]$. Then,

$$\begin{aligned}
\phi(T_{i+1}) &= \phi(T_i a) && \text{(by the definitions of } T_{i+1} \text{ and } a\text{)} \\
&= \delta(\phi(T_i), a) && \text{(by the definition of } \phi\text{)} \\
&= \delta(q, a) && \text{(by the definition of } q\text{)} \\
&= \sigma(P_q a) && \text{(by the definition (32.4) of } \delta\text{)} \\
&= \sigma(T_i a) && \text{(by Lemma 32.3 and induction)} \\
&= \sigma(T_{i+1}) && \text{(by the definition of } T_{i+1}\text{)} \ . \qquad \blacksquare
\end{aligned}$$

By Theorem 32.4, if the machine enters state q on line 4, then q is the largest value such that $P_q \sqsupset T_i$. Thus, we have $q = m$ on line 5 if and only if the machine has just scanned an occurrence of the pattern P. We conclude that FINITE-AUTOMATON-MATCHER operates correctly.

Computing the transition function

The following procedure computes the transition function δ from a given pattern $P[1 \mathinner{.\,.} m]$.

```
COMPUTE-TRANSITION-FUNCTION($P, \Sigma$)
1  $m = P.length$
2  **for** $q = 0$ **to** $m$
3      **for** each character $a \in \Sigma$
4          $k = \min(m + 1, q + 2)$
5          **repeat**
6              $k = k - 1$
7          **until** $P_k \sqsupset P_q a$
8          $\delta(q, a) = k$
9  **return** $\delta$
```

This procedure computes $\delta(q, a)$ in a straightforward manner according to its definition in equation (32.4). The nested loops beginning on lines 2 and 3 consider all states q and all characters a, and lines 4–8 set $\delta(q, a)$ to be the largest k such that $P_k \sqsupset P_q a$. The code starts with the largest conceivable value of k, which is $\min(m, q + 1)$. It then decreases k until $P_k \sqsupset P_q a$, which must eventually occur, since $P_0 = \varepsilon$ is a suffix of every string.

The running time of COMPUTE-TRANSITION-FUNCTION is $O(m^3 |\Sigma|)$, because the outer loops contribute a factor of $m |\Sigma|$, the inner **repeat** loop can run at most $m + 1$ times, and the test $P_k \sqsupset P_q a$ on line 7 can require comparing up to m characters. Much faster procedures exist; by utilizing some cleverly computed information about the pattern P (see Exercise 32.4-8), we can improve the time required to compute δ from P to $O(m\,|\Sigma|)$. With this improved procedure for computing δ, we can find all occurrences of a length-m pattern in a length-n text over an alphabet Σ with $O(m\,|\Sigma|)$ preprocessing time and $\Theta(n)$ matching time.

Exercises

32.3-1
Construct the string-matching automaton for the pattern $P = \texttt{aabab}$ and illustrate its operation on the text string $T = \texttt{aaababaabaababaab}$.

32.3-2
Draw a state-transition diagram for a string-matching automaton for the pattern $\texttt{ababbabbababbababbabb}$ over the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$.

32.3-3
We call a pattern P ***nonoverlappable*** if $P_k \sqsupset P_q$ implies $k = 0$ or $k = q$. Describe the state-transition diagram of the string-matching automaton for a nonoverlappable pattern.

32.3-4 ★
Given two patterns P and P', describe how to construct a finite automaton that determines all occurrences of *either* pattern. Try to minimize the number of states in your automaton.

32.3-5
Given a pattern P containing gap characters (see Exercise 32.1-4), show how to build a finite automaton that can find an occurrence of P in a text T in $O(n)$ matching time, where $n = |T|$.

★ 32.4 The Knuth-Morris-Pratt algorithm

We now present a linear-time string-matching algorithm due to Knuth, Morris, and Pratt. This algorithm avoids computing the transition function δ altogether, and its matching time is $\Theta(n)$ using just an auxiliary function π, which we precompute from the pattern in time $\Theta(m)$ and store in an array $\pi[1 \mathinner{..} m]$. The array π allows us to compute the transition function δ efficiently (in an amortized sense) "on the fly" as needed. Loosely speaking, for any state $q = 0, 1, \ldots, m$ and any character

$a \in \Sigma$, the value $\pi[q]$ contains the information we need to compute $\delta(q, a)$ but that does not depend on a. Since the array π has only m entries, whereas δ has $\Theta(m\,|\Sigma|)$ entries, we save a factor of $|\Sigma|$ in the preprocessing time by computing π rather than δ.

The prefix function for a pattern

The prefix function π for a pattern encapsulates knowledge about how the pattern matches against shifts of itself. We can take advantage of this information to avoid testing useless shifts in the naive pattern-matching algorithm and to avoid precomputing the full transition function δ for a string-matching automaton.

Consider the operation of the naive string matcher. Figure 32.10(a) shows a particular shift s of a template containing the pattern $P = \texttt{ababaca}$ against a text T. For this example, $q = 5$ of the characters have matched successfully, but the 6th pattern character fails to match the corresponding text character. The information that q characters have matched successfully determines the corresponding text characters. Knowing these q text characters allows us to determine immediately that certain shifts are invalid. In the example of the figure, the shift $s + 1$ is necessarily invalid, since the first pattern character (a) would be aligned with a text character that we know does not match the first pattern character, but does match the second pattern character (b). The shift $s' = s + 2$ shown in part (b) of the figure, however, aligns the first three pattern characters with three text characters that must necessarily match. In general, it is useful to know the answer to the following question:

> Given that pattern characters $P[1\,.\,.\,q]$ match text characters $T[s{+}1\,.\,.\,s{+}q]$, what is the least shift $s' > s$ such that for some $k < q$,
>
> $$P[1\,.\,.\,k] = T[s' + 1\,.\,.\,s' + k]\,, \tag{32.6}$$
>
> where $s' + k = s + q$?

In other words, knowing that $P_q \sqsupset T_{s+q}$, we want the longest proper prefix P_k of P_q that is also a suffix of T_{s+q}. (Since $s' + k = s + q$, if we are given s and q, then finding the smallest shift s' is tantamount to finding the longest prefix length k.) We add the difference $q - k$ in the lengths of these prefixes of P to the shift s to arrive at our new shift s', so that $s' = s + (q - k)$. In the best case, $k = 0$, so that $s' = s + q$, and we immediately rule out shifts $s + 1, s + 2, \ldots, s + q - 1$. In any case, at the new shift s' we don't need to compare the first k characters of P with the corresponding characters of T, since equation (32.6) guarantees that they match.

We can precompute the necessary information by comparing the pattern against itself, as Figure 32.10(c) demonstrates. Since $T[s' + 1\,.\,.\,s' + k]$ is part of the

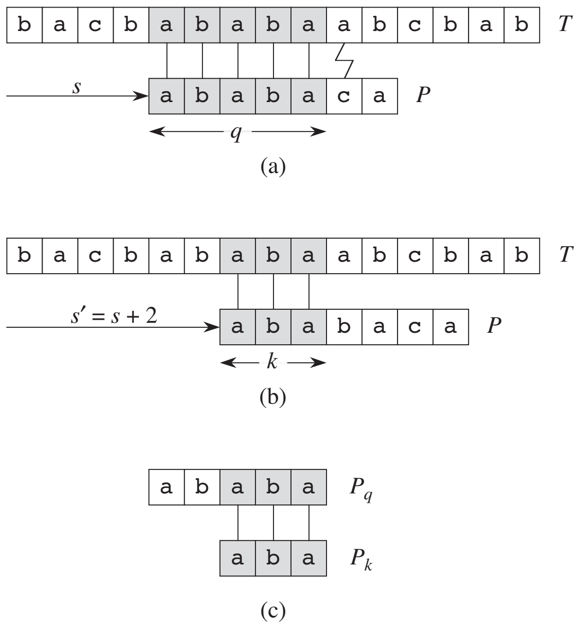


Figure 32.10 The prefix function π. **(a)** The pattern $P =$ `ababaca` aligns with a text T so that the first $q = 5$ characters match. Matching characters, shown shaded, are connected by vertical lines. **(b)** Using only our knowledge of the 5 matched characters, we can deduce that a shift of $s + 1$ is invalid, but that a shift of $s' = s+2$ is consistent with everything we know about the text and therefore is potentially valid. **(c)** We can precompute useful information for such deductions by comparing the pattern with itself. Here, we see that the longest prefix of P that is also a proper suffix of P_5 is P_3. We represent this precomputed information in the array π, so that $\pi[5] = 3$. Given that q characters have matched successfully at shift s, the next potentially valid shift is at $s' = s + (q - \pi[q])$ as shown in part (b).

known portion of the text, it is a suffix of the string P_q. Therefore, we can interpret equation (32.6) as asking for the greatest $k < q$ such that $P_k \sqsupset P_q$. Then, the new shift $s' = s + (q - k)$ is the next potentially valid shift. We will find it convenient to store, for each value of q, the number k of matching characters at the new shift s', rather than storing, say, $s' - s$.

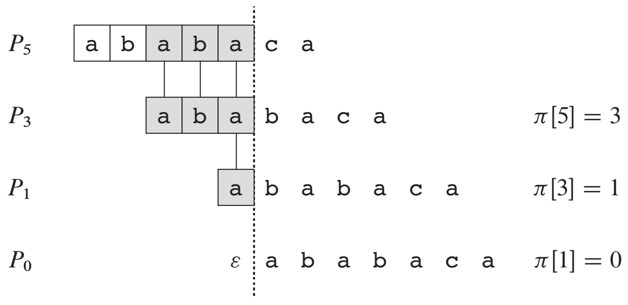
We formalize the information that we precompute as follows. Given a pattern $P[1\,..\,m]$, the ***prefix function*** for the pattern P is the function $\pi : \{1, 2, \ldots, m\} \rightarrow \{0, 1, \ldots, m - 1\}$ such that

$$\pi[q] = \max\{k : k < q \text{ and } P_k \sqsupset P_q\} .$$

That is, $\pi[q]$ is the length of the longest prefix of P that is a proper suffix of P_q. Figure 32.11(a) gives the complete prefix function π for the pattern `ababaca`.

i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

(a)



P_5 a b a b a c a
P_3 a b a b a c a $\pi[5] = 3$
P_1 a b a b a c a $\pi[3] = 1$
P_0 ε a b a b a c a $\pi[1] = 0$

(b)

Figure 32.11 An illustration of Lemma 32.5 for the pattern P = ababaca and $q = 5$. **(a)** The π function for the given pattern. Since $\pi[5] = 3$, $\pi[3] = 1$, and $\pi[1] = 0$, by iterating π we obtain $\pi^*[5] = \{3, 1, 0\}$. **(b)** We slide the template containing the pattern P to the right and note when some prefix P_k of P matches up with some proper suffix of P_5; we get matches when $k = 3$, 1, and 0. In the figure, the first row gives P, and the dotted vertical line is drawn just after P_5. Successive rows show all the shifts of P that cause some prefix P_k of P to match some suffix of P_5. Successfully matched characters are shown shaded. Vertical lines connect aligned matching characters. Thus, $\{k : k < 5 \text{ and } P_k \sqsupset P_5\} = \{3, 1, 0\}$. Lemma 32.5 claims that $\pi^*[q] = \{k : k < q \text{ and } P_k \sqsupset P_q\}$ for all q.

The pseudocode below gives the Knuth-Morris-Pratt matching algorithm as the procedure KMP-MATCHER. For the most part, the procedure follows from FINITE-AUTOMATON-MATCHER, as we shall see. KMP-MATCHER calls the auxiliary procedure COMPUTE-PREFIX-FUNCTION to compute π.

```
KMP-MATCHER($T$, $P$)
 1  $n = T.length$
 2  $m = P.length$
 3  $\pi$ = COMPUTE-PREFIX-FUNCTION($P$)
 4  $q = 0$                                  // number of characters matched
 5  for $i = 1$ to $n$                       // scan the text from left to right
 6      while $q > 0$ and $P[q+1] \neq T[i]$
 7          $q = \pi[q]$                     // next character does not match
 8      if $P[q+1] == T[i]$
 9          $q = q + 1$                      // next character matches
10      if $q == m$                          // is all of $P$ matched?
11          print "Pattern occurs with shift" $i - m$
12          $q = \pi[q]$                     // look for the next match
```

```
COMPUTE-PREFIX-FUNCTION($P$)
 1  $m = P.length$
 2  let $\pi[1\,.\,.\,m]$ be a new array
 3  $\pi[1] = 0$
 4  $k = 0$
 5  for $q = 2$ to $m$
 6      while $k > 0$ and $P[k+1] \neq P[q]$
 7          $k = \pi[k]$
 8      if $P[k+1] == P[q]$
 9          $k = k + 1$
10      $\pi[q] = k$
11  return $\pi$
```

These two procedures have much in common, because both match a string against the pattern P: KMP-MATCHER matches the text T against P, and COMPUTE-PREFIX-FUNCTION matches P against itself.

We begin with an analysis of the running times of these procedures. Proving these procedures correct will be more complicated.

Running-time analysis

The running time of COMPUTE-PREFIX-FUNCTION is $\Theta(m)$, which we show by using the aggregate method of amortized analysis (see Section 17.1). The only tricky part is showing that the **while** loop of lines 6–7 executes $O(m)$ times altogether. We shall show that it makes at most $m-1$ iterations. We start by making some observations about k. First, line 4 starts k at 0, and the only way that k increases is by the increment operation in line 9, which executes at most once per iteration of the **for** loop of lines 5–10. Thus, the total increase in k is at most $m-1$. Second, since $k < q$ upon entering the **for** loop and each iteration of the loop increments q, we always have $k < q$. Therefore, the assignments in lines 3 and 10 ensure that $\pi[q] < q$ for all $q = 1, 2, \ldots, m$, which means that each iteration of the **while** loop decreases k. Third, k never becomes negative. Putting these facts together, we see that the total decrease in k from the **while** loop is bounded from above by the total increase in k over all iterations of the **for** loop, which is $m-1$. Thus, the **while** loop iterates at most $m-1$ times in all, and COMPUTE-PREFIX-FUNCTION runs in time $\Theta(m)$.

Exercise 32.4-4 asks you to show, by a similar aggregate analysis, that the matching time of KMP-MATCHER is $\Theta(n)$.

Compared with FINITE-AUTOMATON-MATCHER, by using π rather than δ, we have reduced the time for preprocessing the pattern from $O(m\,|\Sigma|)$ to $\Theta(m)$, while keeping the actual matching time bounded by $\Theta(n)$.

Correctness of the prefix-function computation

We shall see a little later that the prefix function π helps us simulate the transition function δ in a string-matching automaton. But first, we need to prove that the procedure COMPUTE-PREFIX-FUNCTION does indeed compute the prefix function correctly. In order to do so, we will need to find all prefixes P_k that are proper suffixes of a given prefix P_q. The value of $\pi[q]$ gives us the longest such prefix, but the following lemma, illustrated in Figure 32.11, shows that by iterating the prefix function π, we can indeed enumerate all the prefixes P_k that are proper suffixes of P_q. Let

$$\pi^*[q] = \{\pi[q], \pi^{(2)}[q], \pi^{(3)}[q], \ldots, \pi^{(t)}[q]\} ,$$

where $\pi^{(i)}[q]$ is defined in terms of functional iteration, so that $\pi^{(0)}[q] = q$ and $\pi^{(i)}[q] = \pi[\pi^{(i-1)}[q]]$ for $i \geq 1$, and where the sequence in $\pi^*[q]$ stops upon reaching $\pi^{(t)}[q] = 0$.

Lemma 32.5 (Prefix-function iteration lemma)
Let P be a pattern of length m with prefix function π. Then, for $q = 1, 2, \ldots, m$, we have $\pi^*[q] = \{k : k < q \text{ and } P_k \sqsupset P_q\}$.

Proof We first prove that $\pi^*[q] \subseteq \{k : k < q \text{ and } P_k \sqsupset P_q\}$ or, equivalently,

$$i \in \pi^*[q] \text{ implies } P_i \sqsupset P_q . \tag{32.7}$$

If $i \in \pi^*[q]$, then $i = \pi^{(u)}[q]$ for some $u > 0$. We prove equation (32.7) by induction on u. For $u = 1$, we have $i = \pi[q]$, and the claim follows since $i < q$ and $P_{\pi[q]} \sqsupset P_q$ by the definition of π. Using the relations $\pi[i] < i$ and $P_{\pi[i]} \sqsupset P_i$ and the transitivity of $<$ and $\sqsupset$ establishes the claim for all i in $\pi^*[q]$. Therefore, $\pi^*[q] \subseteq \{k : k < q \text{ and } P_k \sqsupset P_q\}$.

We now prove that $\{k : k < q \text{ and } P_k \sqsupset P_q\} \subseteq \pi^*[q]$ by contradiction. Suppose to the contrary that the set $\{k : k < q \text{ and } P_k \sqsupset P_q\} - \pi^*[q]$ is nonempty, and let j be the largest number in the set. Because $\pi[q]$ is the largest value in $\{k : k < q \text{ and } P_k \sqsupset P_q\}$ and $\pi[q] \in \pi^*[q]$, we must have $j < \pi[q]$, and so we let j' denote the smallest integer in $\pi^*[q]$ that is greater than j. (We can choose $j' = \pi[q]$ if no other number in $\pi^*[q]$ is greater than j.) We have $P_j \sqsupset P_q$ because $j \in \{k : k < q \text{ and } P_k \sqsupset P_q\}$, and from $j' \in \pi^*[q]$ and equation (32.7), we have $P_{j'} \sqsupset P_q$. Thus, $P_j \sqsupset P_{j'}$ by Lemma 32.1, and j is the largest value less than j' with this property. Therefore, we must have $\pi[j'] = j$ and, since $j' \in \pi^*[q]$, we must have $j \in \pi^*[q]$ as well. This contradiction proves the lemma. ■

The algorithm COMPUTE-PREFIX-FUNCTION computes $\pi[q]$, in order, for $q = 1, 2, \ldots, m$. Setting $\pi[1]$ to 0 in line 3 of COMPUTE-PREFIX-FUNCTION is certainly correct, since $\pi[q] < q$ for all q. We shall use the following lemma and its corollary to prove that COMPUTE-PREFIX-FUNCTION computes $\pi[q]$ correctly for $q > 1$.

Lemma 32.6
Let P be a pattern of length m, and let π be the prefix function for P. For $q = 1, 2, \ldots, m$, if $\pi[q] > 0$, then $\pi[q] - 1 \in \pi^*[q-1]$.

Proof Let $r = \pi[q] > 0$, so that $r < q$ and $P_r \sqsupset P_q$; thus, $r - 1 < q - 1$ and $P_{r-1} \sqsupset P_{q-1}$ (by dropping the last character from P_r and P_q, which we can do because $r > 0$). By Lemma 32.5, therefore, $r - 1 \in \pi^*[q-1]$. Thus, we have $\pi[q] - 1 = r - 1 \in \pi^*[q-1]$. ■

For $q = 2, 3, \ldots, m$, define the subset $E_{q-1} \subseteq \pi^*[q-1]$ by

$$\begin{aligned}
E_{q-1} &= \{k \in \pi^*[q-1] : P[k+1] = P[q]\} \\
&= \{k : k < q-1 \text{ and } P_k \sqsupset P_{q-1} \text{ and } P[k+1] = P[q]\} \text{ (by Lemma 32.5)} \\
&= \{k : k < q-1 \text{ and } P_{k+1} \sqsupset P_q\} \; .
\end{aligned}$$

The set E_{q-1} consists of the values $k < q-1$ for which $P_k \sqsupset P_{q-1}$ and for which, because $P[k+1] = P[q]$, we have $P_{k+1} \sqsupset P_q$. Thus, E_{q-1} consists of those values $k \in \pi^*[q-1]$ such that we can extend P_k to P_{k+1} and get a proper suffix of P_q.

Corollary 32.7
Let P be a pattern of length m, and let π be the prefix function for P. For $q = 2, 3, \ldots, m$,

$$\pi[q] = \begin{cases} 0 & \text{if } E_{q-1} = \emptyset \; , \\ 1 + \max\{k \in E_{q-1}\} & \text{if } E_{q-1} \neq \emptyset \; . \end{cases}$$

Proof If E_{q-1} is empty, there is no $k \in \pi^*[q-1]$ (including $k = 0$) for which we can extend P_k to P_{k+1} and get a proper suffix of P_q. Therefore $\pi[q] = 0$.

If E_{q-1} is nonempty, then for each $k \in E_{q-1}$ we have $k+1 < q$ and $P_{k+1} \sqsupset P_q$. Therefore, from the definition of $\pi[q]$, we have

$$\pi[q] \geq 1 + \max\{k \in E_{q-1}\} \; . \tag{32.8}$$

Note that $\pi[q] > 0$. Let $r = \pi[q] - 1$, so that $r + 1 = \pi[q]$ and therefore $P_{r+1} \sqsupset P_q$. Since $r + 1 > 0$, we have $P[r+1] = P[q]$. Furthermore, by Lemma 32.6, we have $r \in \pi^*[q-1]$. Therefore, $r \in E_{q-1}$, and so $r \leq \max\{k \in E_{q-1}\}$ or, equivalently,

$$\pi[q] \leq 1 + \max\{k \in E_{q-1}\} \; . \tag{32.9}$$

Combining equations (32.8) and (32.9) completes the proof. ■

We now finish the proof that COMPUTE-PREFIX-FUNCTION computes π correctly. In the procedure COMPUTE-PREFIX-FUNCTION, at the start of each iteration of the **for** loop of lines 5–10, we have that $k = \pi[q-1]$. This condition is enforced by lines 3 and 4 when the loop is first entered, and it remains true in each successive iteration because of line 10. Lines 6–9 adjust k so that it becomes the correct value of $\pi[q]$. The **while** loop of lines 6–7 searches through all values $k \in \pi^*[q-1]$ until it finds a value of k for which $P[k+1] = P[q]$; at that point, k is the largest value in the set E_{q-1}, so that, by Corollary 32.7, we can set $\pi[q]$ to $k+1$. If the **while** loop cannot find a $k \in \pi^*[q-1]$ such that $P[k+1] = P[q]$, then k equals 0 at line 8. If $P[1] = P[q]$, then we should set both k and $\pi[q]$ to 1; otherwise we should leave k alone and set $\pi[q]$ to 0. Lines 8–10 set k and $\pi[q]$ correctly in either case. This completes our proof of the correctness of COMPUTE-PREFIX-FUNCTION.

Correctness of the Knuth-Morris-Pratt algorithm

We can think of the procedure KMP-MATCHER as a reimplemented version of the procedure FINITE-AUTOMATON-MATCHER, but using the prefix function π to compute state transitions. Specifically, we shall prove that in the ith iteration of the **for** loops of both KMP-MATCHER and FINITE-AUTOMATON-MATCHER, the state q has the same value when we test for equality with m (at line 10 in KMP-MATCHER and at line 5 in FINITE-AUTOMATON-MATCHER). Once we have argued that KMP-MATCHER simulates the behavior of FINITE-AUTOMATON-MATCHER, the correctness of KMP-MATCHER follows from the correctness of FINITE-AUTOMATON-MATCHER (though we shall see a little later why line 12 in KMP-MATCHER is necessary).

Before we formally prove that KMP-MATCHER correctly simulates FINITE-AUTOMATON-MATCHER, let's take a moment to understand how the prefix function π replaces the δ transition function. Recall that when a string-matching automaton is in state q and it scans a character $a = T[i]$, it moves to a new state $\delta(q, a)$. If $a = P[q+1]$, so that a continues to match the pattern, then $\delta(q, a) = q + 1$. Otherwise, $a \neq P[q+1]$, so that a does not continue to match the pattern, and $0 \leq \delta(q, a) \leq q$. In the first case, when a continues to match, KMP-MATCHER moves to state $q + 1$ without referring to the π function: the **while** loop test in line 6 comes up false the first time, the test in line 8 comes up true, and line 9 increments q.

The π function comes into play when the character a does not continue to match the pattern, so that the new state $\delta(q, a)$ is either q or to the left of q along the spine of the automaton. The **while** loop of lines 6–7 in KMP-MATCHER iterates through the states in $\pi^*[q]$, stopping either when it arrives in a state, say q', such that a matches $P[q'+1]$ or q' has gone all the way down to 0. If a matches $P[q'+1]$, then line 9 sets the new state to $q'+1$, which should equal $\delta(q,a)$ for the simulation to work correctly. In other words, the new state $\delta(q,a)$ should be either state 0 or one greater than some state in $\pi^*[q]$.

Let's look at the example in Figures 32.7 and 32.11, which are for the pattern $P = \texttt{ababaca}$. Suppose that the automaton is in state $q = 5$; the states in $\pi^*[5]$ are, in descending order, 3, 1, and 0. If the next character scanned is c, then we can easily see that the automaton moves to state $\delta(5, \texttt{c}) = 6$ in both FINITE-AUTOMATON-MATCHER and KMP-MATCHER. Now suppose that the next character scanned is instead b, so that the automaton should move to state $\delta(5, \texttt{b}) = 4$. The **while** loop in KMP-MATCHER exits having executed line 7 once, and it arrives in state $q' = \pi[5] = 3$. Since $P[q'+1] = P[4] = \texttt{b}$, the test in line 8 comes up true, and KMP-MATCHER moves to the new state $q'+1 = 4 = \delta(5, \texttt{b})$. Finally, suppose that the next character scanned is instead a, so that the automaton should move to state $\delta(5, \texttt{a}) = 1$. The first three times that the test in line 6 executes, the test comes up true. The first time, we find that $P[6] = \texttt{c} \neq \texttt{a}$, and KMP-MATCHER moves to state $\pi[5] = 3$ (the first state in $\pi^*[5]$). The second time, we find that $P[4] = \texttt{b} \neq \texttt{a}$ and move to state $\pi[3] = 1$ (the second state in $\pi^*[5]$). The third time, we find that $P[2] = \texttt{b} \neq \texttt{a}$ and move to state $\pi[1] = 0$ (the last state in $\pi^*[5]$). The **while** loop exits once it arrives in state $q' = 0$. Now, line 8 finds that $P[q'+1] = P[1] = \texttt{a}$, and line 9 moves the automaton to the new state $q'+1 = 1 = \delta(5, \texttt{a})$.

Thus, our intuition is that KMP-MATCHER iterates through the states in $\pi^*[q]$ in decreasing order, stopping at some state q' and then possibly moving to state $q'+1$. Although that might seem like a lot of work just to simulate computing $\delta(q,a)$, bear in mind that asymptotically, KMP-MATCHER is no slower than FINITE-AUTOMATON-MATCHER.

We are now ready to formally prove the correctness of the Knuth-Morris-Pratt algorithm. By Theorem 32.4, we have that $q = \sigma(T_i)$ after each time we execute line 4 of FINITE-AUTOMATON-MATCHER. Therefore, it suffices to show that the same property holds with regard to the **for** loop in KMP-MATCHER. The proof proceeds by induction on the number of loop iterations. Initially, both procedures set q to 0 as they enter their respective **for** loops for the first time. Consider iteration i of the **for** loop in KMP-MATCHER, and let q' be state at the start of this loop iteration. By the inductive hypothesis, we have $q' = \sigma(T_{i-1})$. We need to show that $q = \sigma(T_i)$ at line 10. (Again, we shall handle line 12 separately.)

When we consider the character $T[i]$, the longest prefix of P that is a suffix of T_i is either $P_{q'+1}$ (if $P[q'+1] = T[i]$) or some prefix (not necessarily proper, and possibly empty) of $P_{q'}$. We consider separately the three cases in which $\sigma(T_i) = 0$, $\sigma(T_i) = q'+1$, and $0 < \sigma(T_i) \leq q'$.

- If $\sigma(T_i) = 0$, then $P_0 = \varepsilon$ is the only prefix of P that is a suffix of T_i. The **while** loop of lines 6–7 iterates through the values in $\pi^*[q']$, but although $P_q \sqsupset T_i$ for every $q \in \pi^*[q']$, the loop never finds a q such that $P[q+1] = T[i]$. The loop terminates when q reaches 0, and of course line 9 does not execute. Therefore, $q = 0$ at line 10, so that $q = \sigma(T_i)$.
- If $\sigma(T_i) = q' + 1$, then $P[q'+1] = T[i]$, and the **while** loop test in line 6 fails the first time through. Line 9 executes, incrementing q so that afterward we have $q = q' + 1 = \sigma(T_i)$.
- If $0 < \sigma(T_i) \le q'$, then the **while** loop of lines 6–7 iterates at least once, checking in decreasing order each value $q \in \pi^*[q']$ until it stops at some $q < q'$. Thus, P_q is the longest prefix of $P_{q'}$ for which $P[q+1] = T[i]$, so that when the **while** loop terminates, $q + 1 = \sigma(P_{q'}T[i])$. Since $q' = \sigma(T_{i-1})$, Lemma 32.3 implies that $\sigma(T_{i-1}T[i]) = \sigma(P_{q'}T[i])$. Thus, we have

$$\begin{aligned} q+1 &= \sigma(P_{q'}T[i]) \\ &= \sigma(T_{i-1}T[i]) \\ &= \sigma(T_i) \end{aligned}$$

  when the **while** loop terminates. After line 9 increments q, we have $q = \sigma(T_i)$.

Line 12 is necessary in KMP-MATCHER, because otherwise, we might reference $P[m+1]$ on line 6 after finding an occurrence of P. (The argument that $q = \sigma(T_{i-1})$ upon the next execution of line 6 remains valid by the hint given in Exercise 32.4-8: $\delta(m, a) = \delta(\pi[m], a)$ or, equivalently, $\sigma(Pa) = \sigma(P_{\pi[m]}a)$ for any $a \in \Sigma$.) The remaining argument for the correctness of the Knuth-Morris-Pratt algorithm follows from the correctness of FINITE-AUTOMATON-MATCHER, since we have shown that KMP-MATCHER simulates the behavior of FINITE-AUTOMATON-MATCHER.

Exercises

32.4-1

Compute the prefix function π for the pattern `ababbabbabbababbabb`.

32.4-2

Give an upper bound on the size of $\pi^*[q]$ as a function of q. Give an example to show that your bound is tight.

32.4-3

Explain how to determine the occurrences of pattern P in the text T by examining the π function for the string PT (the string of length $m+n$ that is the concatenation of P and T).

32.4-4

Use an aggregate analysis to show that the running time of KMP-MATCHER is $\Theta(n)$.

32.4-5

Use a potential function to show that the running time of KMP-MATCHER is $\Theta(n)$.

32.4-6

Show how to improve KMP-MATCHER by replacing the occurrence of π in line 7 (but not line 12) by π', where π' is defined recursively for $q = 1, 2, \ldots, m-1$ by the equation

$$\pi'[q] = \begin{cases} 0 & \text{if } \pi[q] = 0 , \\ \pi'[\pi[q]] & \text{if } \pi[q] \neq 0 \text{ and } P[\pi[q]+1] = P[q+1] , \\ \pi[q] & \text{if } \pi[q] \neq 0 \text{ and } P[\pi[q]+1] \neq P[q+1] . \end{cases}$$

Explain why the modified algorithm is correct, and explain in what sense this change constitutes an improvement.

32.4-7

Give a linear-time algorithm to determine whether a text T is a cyclic rotation of another string T'. For example, `arc` and `car` are cyclic rotations of each other.

32.4-8 ★

Give an $O(m\,|\Sigma|)$-time algorithm for computing the transition function δ for the string-matching automaton corresponding to a given pattern P. (*Hint:* Prove that $\delta(q, a) = \delta(\pi[q], a)$ if $q = m$ or $P[q+1] \neq a$.)

Problems

32-1 String matching based on repetition factors

Let y^i denote the concatenation of string y with itself i times. For example, $(\texttt{ab})^3 = \texttt{ababab}$. We say that a string $x \in \Sigma^*$ has ***repetition factor*** r if $x = y^r$ for some string $y \in \Sigma^*$ and some $r > 0$. Let $\rho(x)$ denote the largest r such that x has repetition factor r.

a. Give an efficient algorithm that takes as input a pattern $P[1\,..\,m]$ and computes the value $\rho(P_i)$ for $i = 1, 2, \ldots, m$. What is the running time of your algorithm?

b. For any pattern $P[1\,..\,m]$, let $\rho^*(P)$ be defined as $\max_{1 \le i \le m} \rho(P_i)$. Prove that if the pattern P is chosen randomly from the set of all binary strings of length m, then the expected value of $\rho^*(P)$ is $O(1)$.

c. Argue that the following string-matching algorithm correctly finds all occurrences of pattern P in a text $T[1\,..\,n]$ in time $O(\rho^*(P)n + m)$:

```
REPETITION-MATCHER(P, T)
 1  $m = P.length$
 2  $n = T.length$
 3  $k = 1 + \rho^*(P)$
 4  $q = 0$
 5  $s = 0$
 6  while $s \le n - m$
 7      if $T[s + q + 1] == P[q + 1]$
 8          $q = q + 1$
 9          if $q == m$
10              print "Pattern occurs with shift" $s$
11      if $q == m$ or $T[s + q + 1] \ne P[q + 1]$
12          $s = s + \max(1, \lceil q/k \rceil)$
13          $q = 0$
```

This algorithm is due to Galil and Seiferas. By extending these ideas greatly, they obtained a linear-time string-matching algorithm that uses only $O(1)$ storage beyond what is required for P and T.

Chapter notes

The relation of string matching to the theory of finite automata is discussed by Aho, Hopcroft, and Ullman [5]. The Knuth-Morris-Pratt algorithm [214] was invented independently by Knuth and Pratt and by Morris; they published their work jointly. Reingold, Urban, and Gries [294] give an alternative treatment of the Knuth-Morris-Pratt algorithm. The Rabin-Karp algorithm was proposed by Karp and Rabin [201]. Galil and Seiferas [126] give an interesting deterministic linear-time string-matching algorithm that uses only $O(1)$ space beyond that required to store the pattern and text.